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CALCULATION OF PFAFFIANS BY A CHIP REMOVAL

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We describe a new combinatorial-algebraic transformation on graphs which we call "chip removal." It generalizes the well-known Urban Renewal trick of Propp and Kuperberg. The chip removal is useful in calculations of determinants of adjacency matrices and matching numbers of graphs. A beautiful example of this technique is a theorem on removing four-contact chips, which generalizes Kuo's graphical condensation method. Numerous examples are given. Bibliography: 10 titles.

1. Introduction. Let G be an arbitrary (undirected) graph. Consider an arbitrary orientation of its edges. In this paper, we suggest a combinatorial technique for calculating the Pfaffian $\mathrm{Pf}(G)$, which generalizes the "Urban Renewal" trick of Kuperberg and Propp for counting the number of matchings in a graph and the chip removal technique developed by the authors in [1]. Both approaches are special cases of the diagonalization of block matrices, and they have a very transparent combinatorial interpretation.

We calculate the Pfaffian $\mathrm{Pf}(G)$ by means of a special operation, the *chip removal*. By a *chip* H we mean an arbitrary induced subgraph of G with an even number of vertices. A vertex of a chip that has an outgoing edge (i.e., an edge whose second endpoint lies outside H) will be called *external*, and the second endpoint of an external edge will be called a *contact*. The chip removal operation consists of two steps: 1) we remove the chip H and all its external edges from the graph, and 2) after that, we "repair" the remaining part of the graph by joining some contacts with new weighted edges which we call *jumpers*. The location and weights of the jumpers depend on the chip. Denote by G' the graph obtained by this operation. The main property of the chip removal is that

$$\mathrm{Pf}(G) = \mathrm{Pf}(H)\,\mathrm{Pf}(G').$$

In Sec. 2, we give necessary background on Pfaffians and describe the general scheme of chip removal in terms of the antisymmetric adjacency matrix of the graph. In Sec. 3, we describe the chip removal operation in terms of Pfaffians. In Sec. 4, we give examples of counting the number of matchings in graphs of the form $G \times P_n$ by means of the chip removal technique. As a corollary, we obtain an assertion on the number of matchings in a rectangle. In Sec. 5, we apply this technique to count the number of matchings in graphs on the hexagonal lattice and describe the remarkable "Arnold's snakes," graphs for which the numbers of matchings are the Euler–Bernoulli numbers.

2. The general scheme of chip removal. We need to consider the technical details of the definition of the Pfaffian. Below is a collection of known facts and definitions; for details, see Fulmek's article [6].

1. Let $W = (w_{ij})_{1\le i<j\le 2n}$ be a given triangular array of numbers. Let

$$\mu = \{(i_1, i_2), \dots, (i_{2n-1}, i_{2n})\}$$

be an arbitrary matching (= splitting into pairs) of the set $\{1, 2, \dots, 2n\}$ (in each pair, the smaller number should be written first). Then the sign $\mathrm{sgn}(\mu)$ is, by definition, the sign of the permutation $(i_1 i_2 i_3 i_4 \dots i_{2n-1} i_{2n})$, the weigth $w(\mu)$ of M is given by the formula $w(\mu) =$

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$w_{i_1,i_2} w_{i_3,i_4} \ldots w_{i_{2n-1},i_{2n}}$, and the Pfaffian $\operatorname{Pf} W$ is equal to $\operatorname{Pf} W = \sum \operatorname{sgn}(\mu) w(\mu)$, where the summation runs over all matchings μ of the set $\{1, 2, \ldots, 2n\}$.

2. In the above definition, we can replace the set $\{1, 2, \ldots, 2n\}$ by an arbitrary ordered set with an even number of elements. Let $\mu = \{(i_1, i_2), \ldots, (i_{2n-1}, i_{2n})\}$ be an arbitrary matching of the set $\{1, 2, \ldots, 2n\}$, and let μ' be the matching obtained from μ by removing one pair (i_k, i_{k+1}) with the subsequent shift of indices. Then $\operatorname{sgn} \mu = (-1)^{i_k + i_{k+1} + 1} \operatorname{sgn} \mu'$.

3. Now let G be an arbitrary weighted graph with an even number of vertices, indexed by $1, 2, \ldots, 2n$. Denote by w'_{ij} the weight of an edge (v_i, v_j). We may think that G is the complete graph on $2n$ vertices with some edges having weight 0. Choose an arbitrary orientation of the edges of G and consider the corresponding antisymmetric adjacency matrix $\widetilde{A}(G) = (w_{ij})$. Thus we have $w_{ij} = \pm w'_{ij}$ depending on the orientation of the edge $v_i v_j$. Take the upper triangular part of the matrix $\widetilde{A}(G)$ as an array W. By definition, $\operatorname{Pf}(G) = \operatorname{Pf} \widetilde{A}(G) = \operatorname{Pf} W$.

4. The following formula (Cayley's theorem) holds:

$$\det \widetilde{A}(G) = (\operatorname{Pf} \widetilde{A}(G))^2. \tag{1}$$

This can be proved by constructing a bijection between the summands of the form $\operatorname{sgn}(\sigma) \cdot \alpha_{1,\sigma(1)} \alpha_{2,\sigma(2)} \ldots \alpha_{2n,\sigma(2n)}$ in the definition of the determinant and the pairs of matchings $\operatorname{sgn}(\mu) w(\mu) \operatorname{sgn}(\nu) w(\nu)$ appearing in the right-hand side when we write Pfaffians as sums and remove the parentheses. When constructing this bijection, we may assume that the factor $\alpha_{i,\sigma(i)}$ corresponds to the first Pfaffian.

Due to Cayley's formula (1), the calculation of the Pfaffian reduces to the calculation of the determinant of the adjacency matrix. We will adapt the chip removal technique for calculating determinants developed by the authors in [1] to calculating Pfaffians.

For a given graph G, a *chip* is an arbitrary induced subgraph of G with an even number of vertices. A vertex of the chip that has an outgoing edge (i.e., an edge whose second endpoint lies outside H) will be called *external*, and the second endpoint of an external edge will be called a *contact*.

Consider an arbitrary orientation of the edges of the graph G. A typical example is a Pfaffian orientation. Denote by $\widetilde{A}(G)$ the antisymmetric adjacency matrix of the graph G. Let a chip H contain h vertices and have k contacts. Then the matrix $\widetilde{A}(G)$ has a block form:

$$\widetilde{A}(G) = \begin{pmatrix} \widetilde{A}(H) & K & 0 \\ -K^{\mathsf{T}} & L & * \\ 0 & * & * \end{pmatrix}, \tag{2}$$

where K is an $h \times k$ block that encodes the connections of the chip to the contacts, L is a (possibly zero) $k \times k$ block that encodes the edges of the graph G between the contacts, and the stars correspond to other possible edges outside of the chip. Multiplying the matrix $\widetilde{A}(G)$ by $D = \begin{pmatrix} E & 0 & 0 \\ K^{\mathsf{T}} \widetilde{A}(H)^{-1} & E & 0 \\ 0 & 0 & E \end{pmatrix}$ does not change the determinant, hence

$$\det \widetilde{A}(G) = \det D \cdot \widetilde{A}(G) = \det \begin{pmatrix} \widetilde{A}(H) & K & 0 \\ 0 & \widetilde{L} & * \\ 0 & * & * \end{pmatrix} = \det \widetilde{A}(H) \cdot \det \begin{pmatrix} \widetilde{L} & * \\ * & * \end{pmatrix},$$

where

$$\widetilde{L} = L + K^{\mathsf{T}} \widetilde{A}(H)^{-1} K. \tag{3}$$

We interpret the changes in the block L as a "repair," or "installation of jumpers," i.e., creating additional edges between contacts. The weights of these edges are specified in the matrix $-K^\intercal \widetilde{A}(H)^{-1}K$. Denote the graph obtained by the repair by G'. The matrix $\left(\begin{smallmatrix}\widetilde{L} & * \\ * & *\end{smallmatrix}\right)$ is exactly the antisymmetric adjacency matrix of the repaired graph $\widetilde{A}(G')$. Thus

$$\det \widetilde{A}(G) = \det \widetilde{A}(H) \cdot \det \widetilde{A}(G'),$$

and, by Cayley's theorem,

$$\operatorname{Pf} \widetilde{A}(G) = \pm \operatorname{Pf} \widetilde{A}(H) \cdot \operatorname{Pf} \widetilde{A}(G').$$

3. Applications to calculating Pfaffians and matching numbers. In this section, we describe the weights of the jumpers installed during the chip removal operation in combinatorial terms (Theorem 3.4).

Let G be an arbitrary graph with an even number of vertices. In particular, we do not assume that G is bipartite. Consider an arbitrary orientation of the edges of G, and let $\widetilde{A}(G)$ be the antisymmetric adjacency matrix corresponding to this orientation. Assume that the vertices of G are numbered and denote by $\alpha_{k\ell}$ the entries of the matrix $\widetilde{A}(G)$ according to this numbering. Denote by G_{ij} the directed graph obtained from G by removing the vertices v_i and v_j; and by $\widetilde{G}_{ij}$, the graph obtained from G by removing all outgoing edges of the ith vertex and all ingoing edges of the jth vertex. Let $\widetilde{A}_{ij}$ be the matrix obtained from $\widetilde{A}(G)$ by deleting the ith row and jth column.

Lemma 3.1. $|\det \widetilde{A}_{ij}| = |\operatorname{Pf} \widetilde{A}(G) \operatorname{Pf} \widetilde{A}(G_{ij})|$.

Proof. Denote by $\widetilde{A}_{ijji}$ the matrix obtained from $\widetilde{A}(G)$ by deleting the ith and jth rows and the ith and jth columns. Apply the Dodgson condensation formula for determinants:

$$\det \widetilde{A}(G) \det \widetilde{A}_{ijji} = \det \widetilde{A}_{ii} \det \widetilde{A}_{jj} - \det \widetilde{A}_{ij} \det \widetilde{A}_{ji}.$$

Here $\det \widetilde{A}_{ij} = -\det \widetilde{A}_{ji}$ since the matrix $\widetilde{A}(G)$ is antisymmetric; and $\det \widetilde{A}_{ii} = \det \widetilde{A}_{jj} = 0$ since these are antisymmetric matrices of odd order. Expressing the determinants in the left-hand side by formula (1), we obtain the desired equality. □

In the following theorem, we remove the absolute value signs in the assertion of Lemma 3.1.

Theorem 3.2. *For $i < j$, we have* $\det \widetilde{A}_{ij} = -\operatorname{Pf} \widetilde{A}(G) \operatorname{Pf} \widetilde{A}(G_{ij})$.

Proof. Write the determinant $\det \widetilde{A}_{ij}$ in the form

$$(-1)^{i+j}\alpha_{ij} \det \widetilde{A}_{ij} = \sum_{\sigma} \operatorname{sgn}(\sigma)\alpha_{1,\sigma(1)}\alpha_{2,\sigma(2)} \dots \alpha_{2n,\sigma(2n)},$$

where the sum runs over the elements of the symmetric group S_{2n} for which $\sigma(i) = j$ (so each summand in the right-hand side contains the factor α_{ij}). Using the bijection from the proof of Cayley's theorem, we can rewrite the sum in the form

$$\sum_{\sigma} \operatorname{sgn}(\sigma)\alpha_{1,\sigma(1)}\alpha_{2,\sigma(2)} \dots \alpha_{2n,\sigma(2n)} = \sum_{\mu,\nu} (\operatorname{sgn}(\mu) \cdot w(\mu)) \cdot (\operatorname{sgn}(\nu) \cdot w(\nu)),$$

where the sum runs over pairs of matchings and we may assume that in the right-hand side the factor α_{ij} in each summand corresponds to an edge v_iv_j that belongs to the matching μ. Denote by μ' the matching of the graph G_{ij} obtained by removing the edge v_iv_j from μ. It is clear that every matching of G_{ij} can be obtained in this way from an appropriate (and uniquely determined) matching μ and $\operatorname{sgn} \mu = (-1)^{i+j+1} \operatorname{sgn} \mu'$. Then

$$\sum_{\mu,\nu}(\operatorname{sgn}(\mu) \cdot w(\mu)) \cdot (\operatorname{sgn}(\nu) \cdot w(\nu)) = \sum_{\mu',\nu}(-1)^{j+i+1}\alpha_{ij}(\operatorname{sgn}(\mu')w(\mu')) \cdot (\operatorname{sgn}(\nu) \cdot w(\nu)).$$

Thus

$$(-1)^{i+j}\alpha_{ij}\det\widetilde{A}_{ij} = (-1)^{j+i+1}\alpha_{ij}\sum_{\mu',\nu}(\operatorname{sgn}(\mu')w(\mu'))\cdot(\operatorname{sgn}(\nu)\cdot w(\nu)).$$

Canceling $(-1)^{i+j}\alpha_{ij}$ finishes the proof. $\square$

Theorem 3.3. *Let $B = (b_{ij}) = \widetilde{A}(G)^{-1}$ and $\operatorname{Pf}\widetilde{A}(G) \neq 0$. Then the matrix B is antisymmetric and for $i < j$,*

$$b_{ij} = (-1)^{i+j}\frac{\operatorname{Pf}\widetilde{A}(G_{ij})}{\operatorname{Pf}\widetilde{A}(G)}.$$

Proof. We have $\det\widetilde{A}_{ij} = -\det\widetilde{A}_{ji}$, because the matrices are antisymmetric. Then, by Kramer's formulas and Theorem 3.2,

$$b_{ij} = (-1)^{i+j}\frac{\det\widetilde{A}_{ji}}{\det\widetilde{A}(G)} = (-1)^{i+j}\frac{\operatorname{Pf}\widetilde{A}(G_{ij})\operatorname{Pf}\widetilde{A}(G)}{(\operatorname{Pf}\widetilde{A}(G))^2} = (-1)^{i+j}\frac{\operatorname{Pf}\widetilde{A}(G_{ij})}{\operatorname{Pf}\widetilde{A}(G)}. \qquad \square$$

Remark. It is well known that if v_i and v_j are adjacent vertices in a planar bipartite graph, then the absolute value of the entry b_{ij} of the inverse Kasteleyn matrix is equal to the probability that the random matching contains the domino v_iv_j (see [7]). This assertion is a special case of Theorem 3.3.

Now we can describe the chip removal technique in terms of Pfaffians. Assume for simplicity that every external vertex of the chip is joined with one contact only. Besides, we assume that the vertices of the graph are numbered in such a way that *the numbers of all vertices of the chip are smaller than the numbers of all other vertices of the graph.*

Theorem 3.4. *Let $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ be the external vertices of the chip H and $v_{j_1}, v_{j_2}, \dots, v_{j_k}$ be the corresponding contacts. Let $\operatorname{Pf}\widetilde{A}(G) \neq 0$. For $i_r < i_s$, define the weight of the new jumper between the vertices v_{j_r} and v_{j_s} by the formula*

$$w(v_{j_r}v_{j_s}) = (-1)^{i_r+i_s}w_{i_rj_r}w_{i_sj_s}\frac{\operatorname{Pf}\widetilde{A}(H\setminus\{v_{i_r},v_{i_s}\})}{\operatorname{Pf}\widetilde{A}(H)}. \tag{4}$$

Then the following equalities hold:

$$\det(\widetilde{A}(G)) = \det(\widetilde{A}(H))\cdot\det(\widetilde{A}(G')) \quad \textit{and} \quad \operatorname{Pf}(\widetilde{A}(G)) = \operatorname{Pf}(\widetilde{A}(H))\cdot\operatorname{Pf}(\widetilde{A}(G')).$$

Observe that formula (4), in fact, determines the entry $\widetilde{A}(G')_{j_r,j_s}$ of the antisymmetric matrix $\widetilde{A}(G')$. Therefore, this formula determines the orientation of the new jumpers (though this orientation depends on whether we want the weight to be positive or not).

Proof. The claim follows from the general chip removal scheme (3). Write the matrix $\widetilde{A}(G)$ in the block form (2), let $(b_{ij}) = \widetilde{A}(H)^{-1}$. Since $i_r < j_r$, the entry $\alpha_{j_ri_r}$ is in the block $K^\intercal$ and equals $-w_{i_rj_r}$. The weight of the jumper calculated by formula (3) equals $w_{i_rj_r}w_{i_sj_s}b_{i_ri_s}$. Now we can calculate $b_{i_ri_s}$ by Theorem 3.3 and use formula (1). The formula for determinants is proved.

Due to Cayley's formula (1), we can take the square root and obtain

$$\operatorname{Pf}(\widetilde{A}(G)) = \pm\operatorname{Pf}(\widetilde{A}(H))\cdot\operatorname{Pf}(\widetilde{A}(G')).$$

Let us check that the correct sign here is the "plus" sign. Consider an arbitrary matching of G that is the union of a matching of H and a matching of the remaining graph $G\setminus H$. Assign a large positive weight to each edge of this matching (this weight should be greater than the number of matchings in G), and let the other edges of the graph have weight 1. It suffices to check that the contributions of this matching to both sides of the equality under consideration have the same sign.

Assume that the matching contains edges (i_1, i_2), ..., (i_{2k-1}, i_{2k}) of the chip H and edges (j_1, j_2), ..., (j_{2m-1}, j_{2m}) of the remaining part of the graph. We have $i_p < j_q$ for all p and q, by the restriction on the numbering of the vertices. The sign of each summand in the definition of the Pfaffian $\operatorname{Pf} \widetilde{A}(H)$ is the product of the sign of the permutation $(i_1, i_2, \dots, i_{2k-1}, i_{2k})$ and the signs of the edges imposed by the orientation. The product of the latter signs equals $(-1)^{s(H)}$, where $s(H)$ is the number of edges whose starting point has a greater number than the endpoint. The analogous rule works for the other two Pfaffians. But it is clear that

$$\begin{aligned}\operatorname{sgn}(i_1, i_2, \dots, i_{2k-1}, i_{2k}) \cdot (-1)^{s(H)} \cdot \operatorname{sgn}(j_1, j_2, \dots, j_{2m-1}, i_{2m}) \cdot (-1)^{s(G\setminus H)} \\ = \operatorname{sgn}(i_1, i_2, \dots, i_{2k-1}, i_{2k}, j_1, j_2, \dots, j_{2m-1}, i_{2m}) \cdot (-1)^{s(G)},\end{aligned}$$

and the assertion follows. □

Remark. In the case where the graph G has a Pfaffian orientation (it is a Pfaffian orientation for the chip, too), we have

$$M(G) = \pm M(H) \operatorname{Pf}(G').$$

This equality generalizes a result of Ciucu [4] on the general form of graphical condensation for Pfaffians.

We give an example of an application of this technique. We will consider graphs on the square lattice (the nodes are vertices, the sides of the squares are edges). We will denote the number of matchings in a graph G not only by $M(G)$, but also by the symbol # followed by the schematic representation of the graph.

Theorem 3.5. *Asssume that a $2n \times 2m$ rectangular chip in a graph G has only 4 contacts which are connected to the corner vertices of the chip. Denote by G' the graph obtained by the chip removal and repair (see Fig. 1), with the weights of the new edges equal to*

$$x' = \frac{\#}{\#}, \quad y' = \frac{\#}{\#}, \quad w' = \frac{\#}{\#}, \quad z' = \frac{\#}{\#}.$$

Then

$$M(G) = \# \cdot M(G').$$

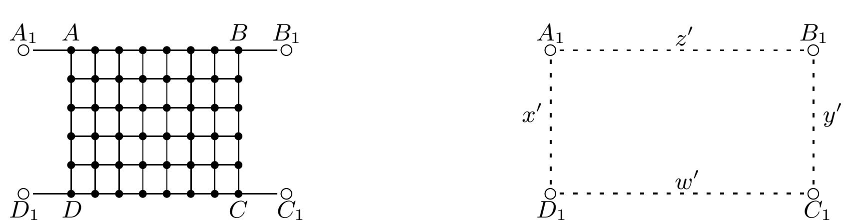


Fig. 1. Removing a 4-contact $2n \times 2m$ chip when counting the number of matchings.

This theorem was proved by the authors in [1] by constructing a bijection. Since the construction uses Kuo's method of graphical condensation, it is essential here that the chip is a planar bipartite graph such that its diagonally opposite corners belong to different parts. Due to this restriction, the number of matchings in the figure is equal to 0. In terms of Theorem 3.4, this means that there will be no jumpers A_1C_1 and B_1D_1 after the removal of the chip. As for the remaining part of the graph G, there are no restrictions at all, this part may be nonbipartite and may have no Pfaffian orientation. However, if the graph does have a Pfaffian orientation, then Theorem 3.5 becomes just a corollary of Theorem 3.4 and formula (4).

4. Examples of the form $G \times P_m$. In the examples, we will use the Fibonacci sequence f_n, where $f_1 = 1$, $f_2 = 2$, and $f_{n+1} = f_n + f_{n-1}$, and the Pell numbers p_n (the sequence A000129 in OEIS):

$$p_1 = 1, \quad p_2 = 2, \quad p_{n+1} = 2p_n + p_{n-1}, \quad p_n = \frac{\sqrt{2}}{4}\Big(\big(1+\sqrt{2}\,\big)^n - \big(1-\sqrt{2}\,\big)^n\Big). \tag{5}$$

Lemma 4.1. (a) *Let x_n be the sequence given by an initial value x_0 and the recurrence*

$$x_{n+1} = x_0 - \frac{1}{x_n}, \quad n \geq 0.$$

Then $x_n = \frac{U_{n+1}(x_0/2)}{U_n(x_0/2)}$, where U_n are Chebyshev polynomials of the second kind.

(b) *The following equalities hold: $f_m = (-i)^{m-1}U_{m-1}(i/2)$, $p_m = (-i)^{m-1}U_{m-1}(i)$.*

(c) *The sequences x_n and y_n given by the initial conditions $x_1 = y_1 = 1$ and recurrences*

$$x_{n+1} = 1 + \frac{1}{x_n}, \qquad y_{n+1} = 2 + \frac{1}{y_n}, \quad n \geq 0,$$

can be given by the explicit formulas $x_n = f_{n+1}/f_n$, $y_n = p_{n+1}/p_n$.

Proof. (a) The Chebyshev polynomials U_m satisfy the recurrence

$$U_{m+1}(x) = 2xU_m - U_{m-1}(x), \qquad U_0 = 1, \quad U_1 = 2x. \tag{6}$$

Since $x_0 = \frac{U_1(x_0/2)}{U_0(x_0/2)}$, we immediately obtain the desired formula.

(b) It is easy to see, due to (6), that the sequences $(-i)^{m-1}U_{m-1}(i/2)$, $(-i)^{m-1}U_{m-1}(i)$ satisfy the recurrences and initial conditions for the Fibonacci and Pell numbers.

(c) Obvious. □

Example 4.1. Let us find the matching numbers of the graphs $W_4 \times P_{m-1}$ and $K_4 \times P_{m-1}$, where W_4 is the 4-cycle with one diagonal and K_4 is the complete graph on 4 vertices. Both graphs are nonbipartite and nonplanar, but they have Pfaffian orientations (see Fig. 2; the same figure depicts also a Pfaffian orientation of $W_4 \times P_{m-1}$ regarded as a subgraph of $K_4 \times P_{m-1}$).

Theorem 4.2. (a) *The matching number of the graph $W_4 \times P_{m-1}$ equals*

$$M(W_4 \times P_{m-1}) = f_m p_m.$$

(b) *The matching number of the graph $K_4 \times P_{m-1}$ equals*

$$M(K_4 \times P_{m-1}) = U_m\Big(\frac{i\sqrt{3}}{2}\Big)U_m\Big(\frac{-i\sqrt{3}}{2}\Big).$$

The first claim of the theorem was proved in [9]. The sequences in the theorem are A001582 and A005386 from OEIS. For even m, the number $|U_m(\frac{i\sqrt{3}}{2})|$ is an integer; for odd m, the number $|\sqrt{3}U_m(\frac{i\sqrt{3}}{2})|$ is an integer. Thus the number $M(K_4 \times P_{m-1})$ is of the form n^2 or $3n^2$ depending on the parity of m. We will prove the theorem by the chip removal technique.

Proof. We will pick and remove chips step by step: $H_1 = A_1B_1C_1D_1$, $H_2 = A_2B_2C_2D_2$, etc. Every chip H_n, $n \geq 2$ (with the jumpers installed after removing the previous chip), is the complete grapf K_4. Consider the removal of this chip in detail. Let the weights of the edges of the chip and of the next "layer" in the graph be given by Fig. 3 (all unmarked edges have weight 1). Then after the chip removal, the new weights are given by the formulas

$$\widetilde{A} = A + \frac{c}{\Delta}, \quad \widetilde{B} = B + \frac{d}{\Delta}, \quad \widetilde{C} = C + \frac{a}{\Delta}, \qquad (7)$$
$$\widetilde{D} = D + \frac{b}{\Delta}, \quad \widetilde{E} = E + \frac{f}{\Delta}, \quad \widetilde{F} = F + \frac{e}{\Delta},$$

where $\Delta = ac+bd+ef = M(H)$. By the way, the fact that all the signs here are pluses proves indirectly that the orientation is Pfaffian.

Consider the matrices

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad S_0 = \begin{pmatrix} 0 & a & f & d \\ -a & 0 & b & -e \\ -f & -b & 0 & c \\ -d & e & -c & 0 \end{pmatrix}, \quad S_0^{-1} = \frac{1}{\Delta}\begin{pmatrix} 0 & -c & -e & -b \\ c & 0 & -d & f \\ e & d & 0 & -a \\ b & -f & -a & 0 \end{pmatrix}.$$

The matrix K here describes the connections of the external edges and contacts of each of the chips H_n (Fig. 3). According to the general chip removal scheme, the rule (7) for calculating the new weights is encoded by the entries of $K^\intercal S_0^{-1} K$ as in (3). Since K is a diagonal matrix with ± 1 on the diagonal, the entries of $K^\intercal S_0^{-1} K$ have the same absolute values as the entries of S_0^{-1} but may have "wrong" signs. This observation allows us to write a reasonable recurrence. Let

$$S_n = S_0 - S_{n-1}^{-1}, \quad n \geq 1; \qquad B_n = \begin{cases} S_n & \text{for even } n, \\ -K^\intercal(S_n)K & \text{for odd } n. \end{cases}$$

We claim that $B_n = \widetilde{A}(H_{n+1})$ for $n \geq 0$. For $n = 0$, this is trivial; for $n = 1$, we have

$$B_1 = -K^\intercal(S_1)K = -K^\intercal(S_0 - S_0^{-1})K = -K^\intercal(S_0)K + K^\intercal(S_0^{-1})K.$$

The matrix $-K^\intercal(S_0)K$ here is exactly the antisymmetric weight matrix of the layer $A_2B_2C_2D_2$, and the second matrix $K^\intercal(S_0^{-1})K$ determines the positions and weights of the jumpers and corresponds to the summands in (7) (now the signs of the entries agree with the orientations of the edges in the layer $A_2B_2C_2D_2$). Thus $B_1 = \widetilde{A}(H_2)$.

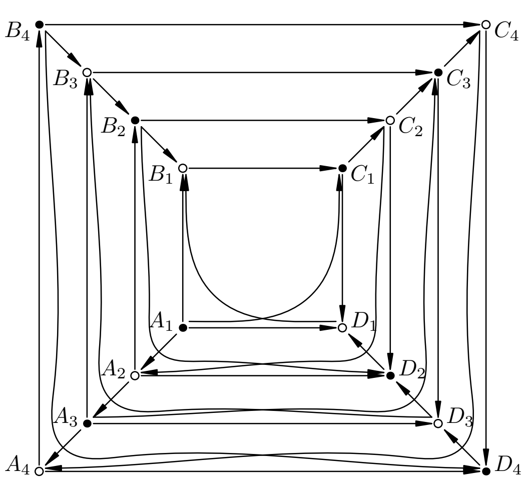


Fig. 2. A Pfaffian orientation of $K_4 \times P_n$.

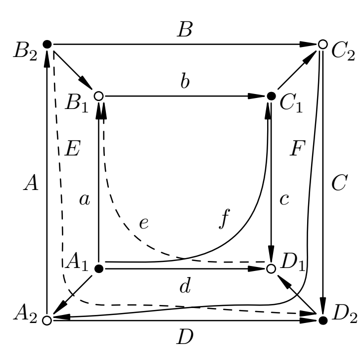


Fig. 3. The chip removal in the graph $K_4 \times P_2$.

At the next layer $A_3B_3C_3D_3$, the antisymmetric adjacency matrix again equals S_0. By the general chip removal scheme, we obtain

$$\widetilde{A}(H_3) = S_0 + K^{\mathsf{T}}(B_1^{-1})K = S_0 - (-K^{\mathsf{T}}B_1K)^{-1} = S_0 - S_1^{-1} = S_2 = B_2.$$

Thus, for $n = 2$ we have $B_2 = \widetilde{A}(H_3)$.

The subsequent operations of removing the nth chip proceed similarly to the case $n = 1$ for odd n, and to the case $n = 2$ for even n.

In order to count the number of matchings in the graph $M(W_4 \times P_{m-1})$, choose the weights $a = b = c = d = f = 1$, $e = 0$. The matrix S_0 given by these weights has the eigenvalues $\pm i$, $\pm 2i$. By Lemma 4.1, we have $S_n = U_{n+1}(S_0/2)U_n(S_0/2)^{-1}$. Then we obtain $\det B_n = \det S_n = \frac{\det U_{n+1}(S_0/2)}{\det U_n(S_0/2)}$ and

$$\begin{aligned} M(W_4 \times P_{m-1}) = \mathrm{Pf}(W_4 \times P_{m-1}) &= \sqrt{\det B_0 \cdot \det B_1 \cdot \ldots \cdot \det B_{m-2}} = \sqrt{\det U_{m-1}(S_0/2)}. \\ &= \sqrt{U_{m-1}(i/2)U_{m-1}(-i/2)U_{m-1}(i)U_{m-1}(-i)} = f_m p_m. \end{aligned}$$

For the graph $K_4 \times P_{m-1}$, we choose the weights $a = b = c = d = f = e = 1$. The eigenvalues of S_0 are $\pm i\sqrt{3}$ with multiplicity 2. Similarly, we have

$$M(K_4 \times P_{m-1}) = \sqrt{\det U_{m-1}(S_0/2)} = U_m\left(\frac{i\sqrt{3}}{2}\right)U_m\left(\frac{-i\sqrt{3}}{2}\right). \qquad \square$$

Example 4.2. Let us calculate the matching number of the cylinder $C_N \times P_m$ for even N and the rectangle $P_{2k} \times P_{m-1}$. As usual, by $T_m(x)$ and $U_m(x)$ we denote the Chebyshev polynomials of the first and second kind, and by p_m we denote the Pell numbers (5).

Lemma 4.3. *Let $N = 2k$ be an even number. Consider the matrices*

$$R_N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \qquad B_N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ -1 & 0 & 1 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -1 & 0 & 0 & \dots & -1 & 0 \end{pmatrix}. \tag{8}$$

Then

(a) *the eigenvalues of the matrix R_N are* $2\cos\frac{\pi\ell}{N+1}$, $\ell = 1, \ldots, N$;

(b) *the eigenvalues of the matrix B_N for $N = 4k+2$ are* $2i\cos\frac{2\pi\ell}{4k+2}$, $\ell = 0, 1, \ldots, 4k+1$, $i = \sqrt{-1}$;

(c) *the eigenvalues of the matrix B_N for $N = 4k$ are* $2i\cos\frac{(2\ell+1)\pi}{4k}$, $\ell = 0, 1, \ldots, 4k-1$.

Proof. This fact is well known and can easily be checked. The characteristic polynomial of the matrix B_N equals $(-1)^{N/2} \cdot 2T_N(\frac{ix}{2}) + 2$, which explains the difference between cases (b) and (c). $\square$

Let

$$K_{k,m} = \left|\prod_{\ell=1}^{k} U_{m-1}\left(i\cos\frac{\pi\ell}{2k+1}\right)\right|.$$

Below we will see (Theorem 4.7) that $K_{k,m}$ is an integer.

Theorem 4.4. (a) *The matching number of the cylinder $C_{4k+2} \times P_{m-1}$ can be represented in the form*

$$M(C_{4k+2} \times P_{m-1}) = p_m K_{k,m}^2.$$

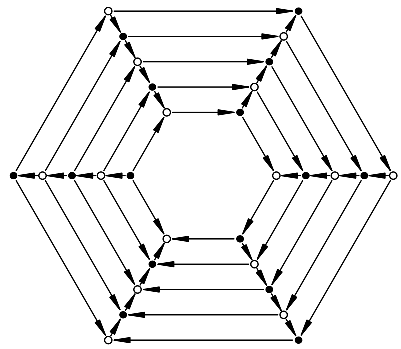


Fig. 4. A Pfaffian orientation of the cylinder $C_{2k} \times P_n$.

(b) *The matching number of the cylinder* $C_{4k} \times P_{m-1}$ *can be represented in the form*

$$M(C_{4k} \times P_{m-1}) = \left(\prod_{\ell=0}^{k-1} U_{m-1}\Big(i \cos \frac{(2\ell+1)\pi}{4k}\Big)\right)^2.$$

(c) *The matching number of the rectangle* $P_{2k} \times P_{m-1}$ *can be represented in the form*

$$M(P_{2k} \times P_{m-1}) = K_{k,m}.$$

Proof. (a), (b) The graph $C_N \times P_m$ is bipartite; color the vertices of its parts in black and white. Choose a Pfaffian orientation as in Fig. 4. As in the previous example, we will remove chips H_n step by step, each of them being a cycle C_N together with the jumpers obtained at the previous step. The first chip H_0 is a cycle C_N at the edge of the cylinder. The antisymmetric adjacency matrix of the N-cycle is given by (8):

$$A_0 = \widetilde{A}(C_N) = B_N;$$

let $A_n = \widetilde{A}(H_n)$ be the antisymmetric adjacency matrix of the next chip, which is a graph on N vertices located at the edge of the cylinder after n steps.

Consider the chip H_n in detail. The weights of its edges "consist of" the unit weights of the edges of the N-cycle and the weights of the jumpers obtained by the removal of the previous chip (for uniformity, we may assume that for $n = 0$ the jumpers have zero weight). The jumpers are diagonal edges that join vertices of different color (this can be proved by induction, due to formula (4): if the contacts v_{j_r}, v_{j_s} are of the same color, then the numerator in the right-hand side vanishes).

The graph under consideration has two specific properties: 1) it is bipartite; 2) all contact edges are directed from a black vertex to a white one (or from a white vertex to a black one, depending on the parity of n), which means that half of the contact edges are directed towards the chip, and another half are directed from the chip. Due to this fact, the jumper matrix $K^{\mathsf{T}}\widetilde{A}(H_n)^{-1}K$ in formula (3) equals $-\widetilde{A}(H_n)^{-1}$. Indeed, let us first number all black vertices in the chip, and then all white vertices, and similarly for the contacts. Then the matrix $\widetilde{A}(H_n)$ has the block form $\left(\begin{smallmatrix} 0 & X_n \\ -X_n & 0 \end{smallmatrix}\right)$, and $K = \left(\begin{smallmatrix} 0 & \pm E \\ \mp E & 0 \end{smallmatrix}\right)$. Comparing the product $K^{\mathsf{T}}\widetilde{A}(H_n)^{-1}K$ with the initial matrix $\widetilde{A}(H_n)^{-1}$, we see that all nonzero entries have changed their sign.

Thus, by formula (3), we have

$$A_{n+1} = A_0 - A_n^{-1}. \tag{9}$$

By Lemma 4.1, we obtain

$$A_n = U_n(A_0/2)U_{n-1}(A_0/2)^{-1}.$$

Hence

$$\det \widetilde{A}(C_N \times P_{m-1}) = \det A_0 \cdot \det A_2 \cdot \ldots \cdot \det A_{m-1} = \det U_{m-1}(A_0/2).$$

We will transform this expression applying Lemma 4.3; the result depends on the parity of $N/2$.

For $N = 4k+2$, the eigenvalues of A_0 are $2i\cos\frac{2\pi\ell}{4k+2}$, $\ell = 0, 1, \ldots, 4k+1$. Each cosine appears twice in this list, except for $\cos\frac{2\pi 0}{4k+2} = 1$ and $\cos\frac{2\pi(2k+1)}{4k+2} = -1$; we may think that the latter two form a pair, since U_{m-1} is an even (or odd) function. The determinant is the product of the eigenvalues, therefore,

$$\begin{aligned} M(C_{4k+2} \times P_{m-1}) &= \sqrt{\det \widetilde{A}(C_{4k+2} \times P_{m-1})} = (-i)^{m-1} \prod_{\ell=0}^{2k} U_{m-1}\Big(i\cos\frac{\pi\ell}{2k+1}\Big) \\ &= (-i)^{m-1} U_{m-1}(i) \left(\prod_{\ell=1}^{2k} U_{m-1}\Big(i\cos\frac{\pi\ell}{2k+1}\Big)\right) = p_m K_{k,m}^2. \end{aligned}$$

For $N = 4k$, the eigenvalues of A_0 are $2i\cos\frac{(2\ell+1)\pi}{4k}$, $\ell = 0, 1, \ldots, 4k-1$. And we have the analogous formula

$$\begin{aligned} M(C_{4k} \times P_{m-1}) &= \sqrt{\det \widetilde{A}(C_{4k} \times P_{m-1})} \\ &= \prod_{\ell=0}^{2k-1} U_{m-1}\Big(i\cos\frac{(2\ell+1)\pi}{4k}\Big) = \left(\prod_{\ell=0}^{k-1} U_{m-1}\Big(i\cos\frac{(2\ell+1)\pi}{4k}\Big)\right)^2. \end{aligned}$$

Below we will check that this number is a perfect square for even m, and twice a perfect square for odd m.

(c) Observe that if we consider the standard Pfaffian orientation of the rectangle, and choose H_n to be the path P_{2k} (plus jumpers) at the edge of the rectangle, then the recurrence (9) is valid for rectangles, too. The eigenvalues of $A_0 = \widetilde{A}(P_{2k})$ are $2i\cos\frac{\pi\ell}{2k+1}$, $\ell = 1, 2, \ldots, 2k$. Taking into account that U_{m-1} is an even/odd function, we have

$$M(P_{2k} \times P_{m-1}) = \sqrt{\det \widetilde{A}(P_{2k} \times P_{m-1})} = \left|\prod_{\ell=1}^{k} U_{m-1}\Big(i\cos\frac{\pi\ell}{2k+1}\Big)\right| = K_{k,m}. \qquad \square$$

Comparing assertions (a) and (c) of the theorem, we obtain a corollary.

Corollary 4.4.1. $M(C_{4k+2} \times P_{m-1}) = p_m M(P_{2k} \times P_{m-1})^2$.

In particular, for $k = 1$, due to Lemma 4.1(b), we have $M(C_6 \times P_{m-1}) = f_m^2 p_m$. This is the sequence A028477 in OEIS, its generating function is calculated in [8].

Now we will factor the numbers $M(P_{2k} \times P_{m-1})$ and $M(C_{4k} \times P_{m-1})$ into integers, which, however, are neither prime nor even coprime.

Recall that the *cyclotomic polynomial* Φ_k is the polynomial whose roots are the primitive kth roots of 1. It is known that the polynomials Φ_k have integer coefficients. The next lemma contains known technical assertions, the keyword here is "the minimal polynomial for $\cos\frac{\pi}{k}$." We have not found a good reference, so give a proof here.

Lemma 4.5. *Let k be an arbitrary nonnegative integer. Then*

(a) *for every nonnegative integer a, the sum* $\sum\limits_{\substack{1\le j\le\lfloor\frac{k}{2}\rfloor,\\ (j,k)=1}} 2^{2a}\cos^{2a}(\frac{j\pi}{k})$ *is an integer;*

(b) $\left|\prod\limits_{\substack{1\le j\le k,\\(j,2k+1)=1}} 2\cos\frac{j\pi}{2k+1}\right| = 1;$ (c) $\prod\limits_{j=0}^{k-1} 2\cos\frac{(2j+1)\pi}{4k} = \sqrt{2}.$

Proof. (a) Using Euler's formula for the cosine, we can rewrite the formula as follows:

$$\sum_{\substack{1\le j\le\lfloor\frac{k}{2}\rfloor,\\(j,k)=1}} 2^{2a}\cos^{2a}\Big(\frac{j\pi}{k}\Big) = \sum_j (e^{i\frac{j\pi}{k}} + e^{-i\frac{j\pi}{k}})^{2a}$$

$$= \sum_j \sum_{m=0}^{2a} \binom{2a}{m} e^{i\frac{(2a-2m)j\pi}{k}} = \sum_m \binom{2a}{m} \sum_{\substack{1\le j\le\lfloor\frac{k}{2}\rfloor,\\(j,k)=1}} e^{i\frac{(a-m)2j\pi}{k}}.$$

The last sum over j is an integer; indeed, if $(a-m,k)=1$, then this is the sum of all primitive kth roots of 1, which is equal to some coefficient of Φ_k; and if $(a-m,k)=d>1$, then the sum contains all primitive (k/d)th roots of 1 with multiplicity d.

(b) Observe that $\left|\prod\limits_{j=1}^{2k} 2\cos\frac{j\pi}{2k+1}\right| = 1$, because this is the constant term of the polynomial $U_{2k}(x/2)$. All the cosines in this product form pairs such that in each pair the cosines differ only by the sign; therefore, $\prod\limits_{j=1}^{k} 2\cos\frac{j\pi}{2k+1} = 1$. Denote this product by $Q(2k+1)$ and the product from the statement of the lemma by $P(2k+1)$. Then for all k,

$$1 = Q(2k+1) = \prod_{j=1}^{k} 2\cos\frac{j\pi}{2k+1} = \prod_{d\mid(2k+1)} \prod_{\substack{1\le j\le k,\\(j,2k+1)=d}} 2\cos\frac{j\pi}{2k+1} = \prod_{d\mid(2k+1)} P\big(\tfrac{2k+1}{d}\big).$$

Now we can obtain an expression for $P(2k+1)$ by the inclusion–exclusion principle: $P(2k+1) = \prod\limits_{d\mid(2k+1)} Q\big(\frac{2k+1}{d}\big)^{\mu(d)} = 1$, where $\mu(d)=0,\pm1$ is the Möbius function.

(c) Observe that $\prod\limits_{j=1}^{k-1}\cos\frac{j\pi}{2k}\cdot\prod\limits_{j=k+1}^{2k-1}\cos\frac{j\pi}{2k}$ is the product of all roots of the polynomial U_{2k-1} except 0; therefore, by the Viète theorem, this product equals $\pm k$ (the coefficient of x in U_{2k-1}). Then

$$\prod_{j=0}^{k-1} 2\cos\frac{(2j+1)\pi}{4k} = \frac{\prod\limits_{j=1}^{2k-1} 2\cos\frac{j\pi}{4k}}{\prod\limits_{j=1}^{k-1} 2\cos\frac{j\pi}{2k}} = \sqrt{\left|\frac{\prod\limits_{j=1}^{2k-1}\cos\frac{j\pi}{4k}\cdot\prod\limits_{j=2k+1}^{4k-1}\cos\frac{j\pi}{4k}}{\prod\limits_{j=1}^{k-1}\cos\frac{j\pi}{2k}\cdot\prod\limits_{j=k+1}^{2k-1}\cos\frac{j\pi}{2k}}\right|} = \sqrt{\frac{2k}{k}} = \sqrt{2}. \qquad \square$$

Lemma 4.6. (a) *Let $x_1, \ldots, x_n$ be real numbers such that for every nonnegative integer k, the sum $\sum\limits_{j=1}^{n} x_j^k$ is an integer. Then for any nonnegative m and any ordered set $(a_1,\ldots,a_m)$ of nonnegative integers, the sum $\sum\limits_{(j_1,\ldots,j_m)}\prod\limits_{k=1}^{m} x_{j_k}^{a_k}$ is an integer (the sum runs over all possible ordered sets of m different numbers $j_1,\ldots,j_m\in\{1,\ldots,m\}$).*

(b) *Let $x_1, \ldots, x_n$ be real numbers such that for every nonnegative integer k, the sum $\sum\limits_{i=1}^{n} x_i^{2k}$ is an integer, and let P be a polynomial with integer coefficients. Then the following holds. If*

P is an even function, then the product $\prod_{i=1}^{n} P(x_i)$ *is an integer; if P is an odd function, then the product* $\prod_{i=1}^{n} \frac{P(x_i)}{x_i}$ *is an integer.*

The first claim can be proved by induction on m. The second claim follows from the first one.

Theorem 4.7. *The following decompositions hold, the value of the product corresponding to every h being an integer for each of them:*

$$M(P_{2k} \times P_{m-1}) = \prod_{h|2k+1} \left| \prod_{\substack{1 \le j \le k, \\ (j,2k+1)=h}} U_{m-1}\left(i \cos \frac{j\pi}{2k+1}\right) \right|, \tag{10}$$

$$M(C_{4k} \times P_{2m}) = \left(\prod_{h|k} \left| \prod_{\substack{0 \le j \le k-1, \\ (2j+1,k)=h}} U_{2m}\left(i \cos \frac{(2j+1)\pi}{4k}\right) \right| \right)^2,$$

$$M(C_{4k} \times P_{2m-1}) = 2 \left(\prod_{h|k} \left| \prod_{\substack{0 \le j \le k-1, \\ (2j+1,k)=h}} \frac{U_{2m-1}\left(i \cos \frac{(2j+1)\pi}{4k}\right)}{2i \cos \frac{(2j+1)\pi}{4k}} \right| \right)^2.$$

Proof. Let us prove formula (10).

For odd m, the polynomial $U_{m-1}(x/2)$ has integer coefficients and is an even function. For every nonnegative integer a and any divisor $h|(2k+1)$, the following equality holds:

$$\sum_{\substack{1 \le j \le k, \\ (j,2k+1)=h}} \left(2i \cos \frac{j\pi}{2k+1}\right)^{2a} = \pm \sum_{\substack{1 \le j \le \lfloor \frac{\widetilde{k}}{2} \rfloor, \\ (j,\widetilde{k})=1}} 2^{2a} \left(\cos \frac{j\pi}{\widetilde{k}}\right)^{2a},$$

where $\widetilde{k} = \frac{2k+1}{h}$. By Lemma 4.5(a), the sum in the right-hand side is an integer. Then, by Lemma 4.6(b), each product over j in formula (10) is an integer.

For even m, the polynomial $U_{m-1}(x/2)$ has integer coefficients and is an odd function. As in the previous paragraph, we obtain by Lemmas 4.5(a) and 4.6(b) that the product $\prod_{\substack{1 \le j \le k, \\ (j,2k+1)=h}} \frac{U_{m-1}\left(i \cos \frac{j\pi}{2k+1}\right)}{2i \cos \frac{j\pi}{2k+1}}$ is an integer. It remains to observe that the product of the denominators equals 1 by Lemma 4.5(b).

The two other formulas can be proved analogously; the factor 2 appears in the last formula due to Lemma 4.5(c). $\square$

From this theorem we can immediately obtain Sellers' [9] observation that the number $M(P_8 \times P_{m-1})$ is divisible by f_m. Indeed, for $k \equiv 1 \pmod 3$, the factor in (10) corresponding to $h = (2k+1)/3$ is the Fibonacci number $f_m = |U_{m-1}(i/2)|$.

The decompositions in the theorem are far from being decompositions into primes or at least coprime integers. For example, $M(P_8 \times P_8) = 12988816 = 2^4 \cdot 17^2 \cdot 53^2$ (see [10]). Applying assertion (a) of the theorem for $k = 4$, $m = 9$, we have for $h = 1$

$$\prod_{\substack{1\le j\le k,\\(j,2k+1)=h}} U_{m-1}\left(i\cos\frac{j\pi}{2k+1}\right) = U_8\left(i\cos\frac{\pi}{9}\right)\cdot U_8\left(i\cos\frac{2\pi}{9}\right)\cdot U_8\left(i\cos\frac{4\pi}{9}\right) = 382024 = 2^3\cdot 17\cdot 53^2,$$

and for $h = 3$

$$\prod_{\substack{1\le j\le k,\\(j,2k+1)=h}} U_{m-1}\left(i\cos\frac{j\pi}{2k+1}\right) = U_8\left(i\cos\frac{\pi}{3}\right) = 34 = 2\cdot 17.$$

5. Examples on the hexagonal lattice

Definition. *A hexagonal $a\times b$ parallelogram* is a figure on the hexagonal lattice consisting of b rows of a hexagons each, where every next row is shifted half the width of the hexagon to the right with respect to the previous row, see Fig. 5. Denote by $B_{a,b}$ the graph whose vertices are the nodes of the parallelogram and edges are the lines of the hexagonal lattice.

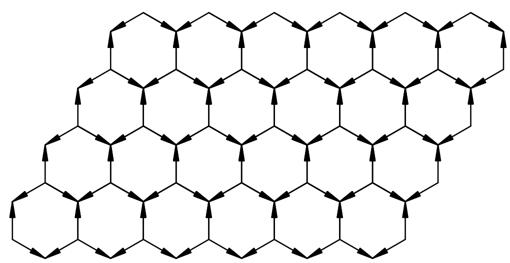


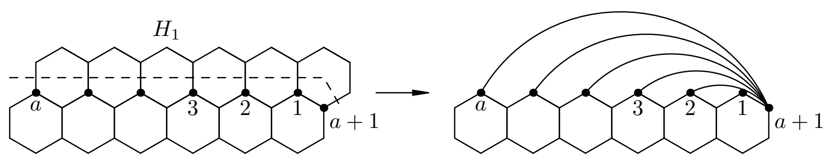
Fig. 5. The hexagonal parallelogram $B_{6,4}$.

The following theorem was proved in [5]. We suggest a proof by means of the chip removal technique.

Theorem 5.1. *The number of matchings in the hexagonal parallelogram $B_{a,b}$ equals $\binom{a+b}{a}$.*

Proof. Fix a Pfaffian orientation of the graph $B_{a,b}$ as in Fig. 5. Assume that the edges of the initial graph have weight 1. We will successively remove chips.

Let the first chip H_1 be the upper "horizontal" layer consisting of $2a + 2$ vertices (in the figure below, the dotted line cuts off the chip, the contacts are numbered). It is easy to see that after the chip removal all the jumpers "share" the vertex $a + 1$. Indeed, if we remove two vertices of the chip connected with the ith and jth contacts (i, $j < a + 1$), then the chip splits into two odd components and has no matchings, so that the weight of the corresponding jumper equals 0. Note that the jumper $(a+1) \to 1$ is not shown in the figure, we assume that its weight is just added to the existing edge.



After the removal of each next chip H_k, we have a similar picture: the jumpers share the rightmost vertex (Fig. 6). Let us calculate the weights of the jumpers after the removal of the chip H_k. Number the vertices of the two consecutive layers as in Fig. 6. Let the weights of the jumpers of the previous chip be equal to $w_{1,a+1}$, $w_{2,a+1}$, $\dots$, $w_{a,a+1}$, and the orientation of the jumpers be "hidden" in the signs of the weights.

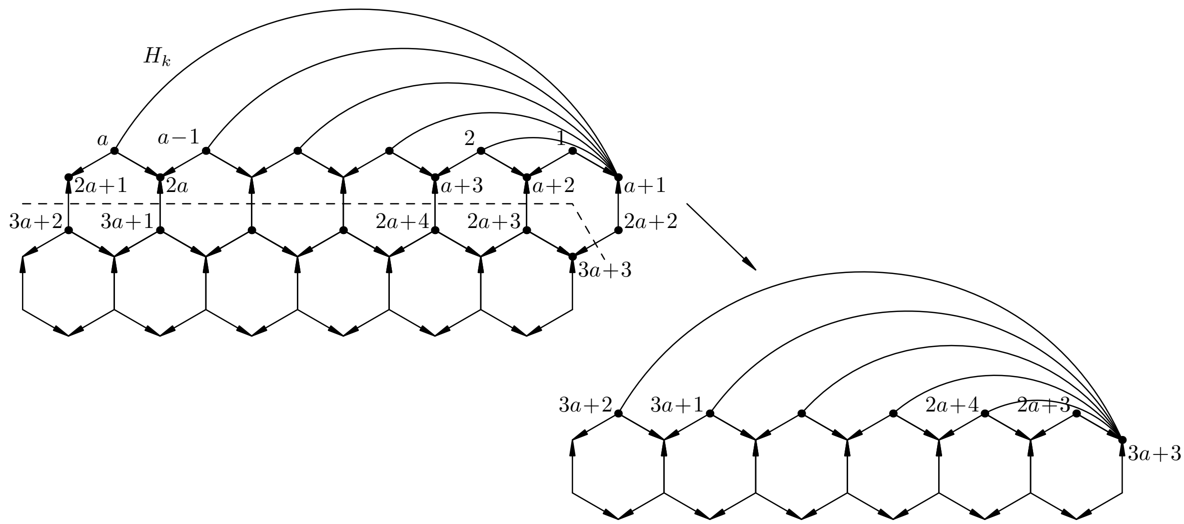


Fig. 6. Removing the chip H_k in the graph $B_{a,b}$.

Since the chip H_k has a unique perfect matching,

$$\begin{aligned}\mathrm{Pf}(H_k) &= \mathrm{sgn}(1, a+2, 2, a+3, \dots, a, 2a+1, a+1, 2a+2)\cdot w_{1,a+2}w_{2,a+3}\cdots w_{a,2a+1}w_{a+1,2a+2}\\ &= (-1)^{\frac{a(a+1)}{2}}\cdot 1\cdot 1\cdot\ldots\cdot 1\cdot(-1) = (-1)^{\frac{a(a+1)}{2}+1}. \end{aligned} \quad (11)$$

Denote by $W_{2a+2+i,3a+3}$ the weight of the jumper that joins the contacts $2a+2+i$ and $3a+3$. Note that for $i = 1$, this jumper is added to the existing edge; in this case, we will use the Kronecker symbol δ_{i1}. In order to calculate the weight $W_{2a+2+i,3a+3}$ by formula (4), we should remove from the chip the second endpoints of the contact edges, i.e., the external vertices $a+1+i$ and $2a+2$, and calculate the Pfaffian of the remaining part of the chip:

$$\begin{aligned}&W_{2a+2+i,3a+3}\\ &\quad= \delta_{i1} + (-1)^{(a+1+i)+(2a+2)}\cdot w_{a+1+i,2a+2+i}\cdot w_{2a+2,3a+3}\cdot\frac{\mathrm{Pf}(H_k\setminus\{a+1+i, 2a+2\})}{\mathrm{Pf}(H_k)}\\ &\quad= \delta_{i1} + (-1)^{a+1+\frac{a(a+1)}{2}+i}\cdot\mathrm{Pf}(H_k\setminus\{a+1+i, 2a+2\}).\end{aligned}$$

It remains to calculate the Pfaffian $H_k \setminus \{a+1+i, 2a+2\}$. The matchings in this graph are uniquely determined by the "arc" edge $(j, a+1)$. Hence

$$\begin{aligned}&\mathrm{Pf}(H_k\setminus\{a+1+i, 2a+2\})\\ &= \sum_{j=1}^{i}\mathrm{sgn}(1, a{+}2, \dots, j{-}1, a{+}j, \underline{j, a{+}1}, j{+}1, a{+}j{+}1, \dots, i, a{+}i, i{+}1, a{+}i{+}2, \dots, a, 2a{+}1)\\ &\qquad\qquad\times w_{1,a+2}\cdot\ldots\cdot w_{a,2a+1}\\ &= \sum_{j=1}^{i}(-1)^{\frac{a(a-1)}{2}-(j-1)}w_{j,a+1}. \end{aligned} \quad (12)$$

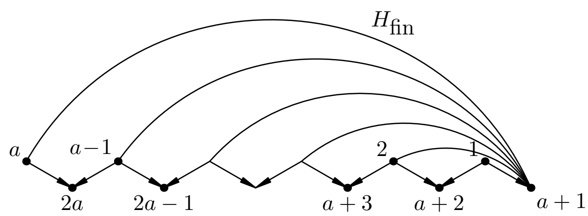


Fig. 7. The result of removing the chip H_b in the graph $B_{a,b}$.

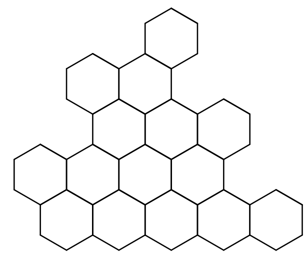


Fig. 8. Arnold's snake of order 5.

Combining the two previous formulas, we obtain

$$W_{2a+2+i,3a+3} = \delta_{i1} + (-1)^{i-1} \cdot \sum_{j=1}^{i} (-1)^{j-1} w_{j,a+1}.$$

Let us change the notation: let $A_{k-1,j} = (-1)^{j-1} w_{j,a+1}$, $j = 1, 2, \ldots, a$. In particular, $A_{0,1} = 1$, $A_{0,2} = A_{0,3} = \ldots = A_{0,a} = 0$ (these equalities mean that the chip H_1 has no jumpers). In the same terms, $W_{2a+2+i,3a+3} = (-1)^{i-1} A_{k,i}$, and we can rewrite the obtained equality in the form

$$A_{k,i} = \delta_{i1} + \sum_{j=1}^{i} A_{k-1,j}.$$

It is not hard to see that the following sequence solves this recurrence:

$$A_{k,j} = \binom{k+j-1}{j} \quad \text{for } j > 1; \qquad A_{k,1} = \binom{k}{1} + 1.$$

After b chip removal operations, almost the whole graph is removed and we obtain the graph H_{fin} (see Fig. 7); its Pfaffian $\operatorname{Pf}(H_{\text{fin}})$ can be calculated similarly to (12):

$$\operatorname{Pf}(H_{\text{fin}}) = (-1)^{\frac{a(a-1)}{2}} \sum_{i=1}^{a} (-1)^{i-1} w_{i,a+1} \tag{13}$$

$$= (-1)^{\frac{a(a-1)}{2}} \sum_{i=1}^{a} A_{b,i} = (-1)^{\frac{a(a-1)}{2}} \sum_{i=1}^{a} \binom{b+i-1}{i} = (-1)^{\frac{a(a-1)}{2}} \binom{b+a}{a}. \tag{14}$$

Thus $M(B_{a,b}) = \left|\operatorname{Pf}(H_1) \cdot \operatorname{Pf}(H_2) \cdot \ldots \cdot \operatorname{Pf}(H_b) \cdot \operatorname{Pf}(H_{\text{fin}})\right| = \binom{b+a}{a}$. □

Definition. Consider the figure on the hexagonal lattice consisting of $\frac{n(n+1)}{2}$ hexagons arranged in n rows such that the ith row contains i hexagons and the rows are shifted to the left or the right as shown in Fig. 8. We will call this figure, as well as the graph S_n formed by the nodes and line segments in it, *Arnold's snake of order n.*

Recall the construction of the Euler–Bernoulli triangle (see [2]). This triangle looks like the Pascal triangle. The top vertex of the triangle (this is the 0th row) contains 1. Every element t in an odd row equals the sum of the elements in the previous row to the left of t, and every element t in an even row equals the sum of the elements in the previous row to the right of t.

Let us number the elements from left to right in odd rows, and from right to left in even rows:

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 0 && 1 &&&& \\ &&& 1 && 1 && 0 &&& \\ && 0 && 1 && 2 && 2 && \\ & 5 && 5 && 4 && 2 && 0 & \\ 0 && 5 && 10 && 14 && 16 && 16 \end{array} \qquad \begin{array}{ccccccc} &&& t_{00} &&& \\ && t_{10} && t_{11} && \\ & t_{22} && t_{21} && t_{20} & \\ t_{30} && t_{31} && t_{32} && t_{33} \\ &&& \dots &&& \\ &&& \dots &&& \end{array} \tag{15}$$

With this numbering, the recurrence for the Euler–Bernoulli triangle takes the form

$$t_{n,0} = 0, \qquad t_{n,i} = \sum_{j=n-i}^{n-1} t_{n-1,j}.$$

The sequence $\{t_{n,n}\}_{n\geq 1}$, that starts as $1, 1, 2, 5, 16, \dots$, is called the *Euler–Bernoulli sequence*. The nonzero numbers $E_n = t_{2n,2n}$ at the left side of the triangle are the Euler numbers, $\sum \frac{E_n}{(2n)!} t^{2n} = \sec t$; and the nonzero numbers $T_n = t_{2n-1,2n-1}$ at the right side are the tangent numbers, $\sum \frac{T_n}{(2n-1)!} t^{2n-1} = \tan t$.

Theorem 5.2. *The matching number of Arnold's snake of order n equals the $(n+2)$th Euler–Bernoilli number $t_{n+2\,n+2}$.*

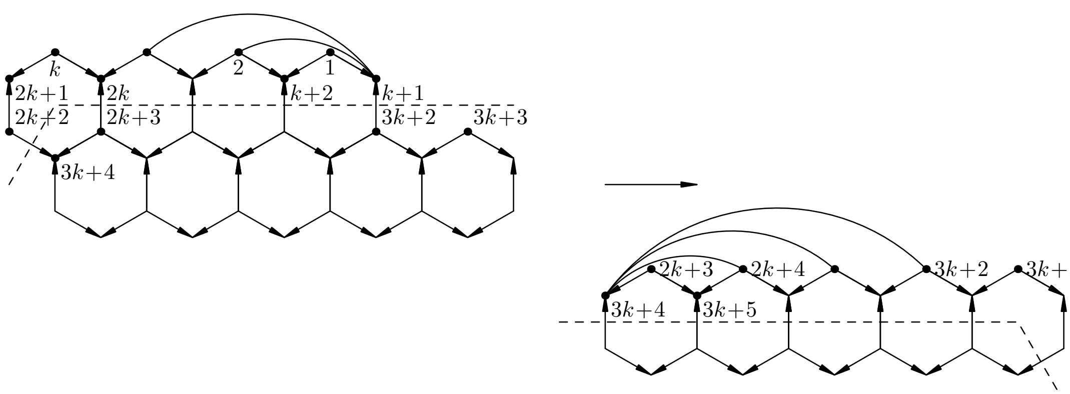
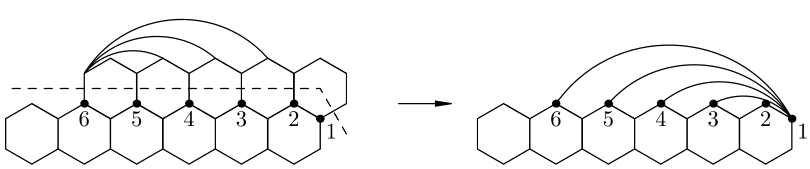


Fig. 9. Removing the chip H_k in Arnold's snake.

Proof. We will remove chips from top to bottom, each chip containing all vertices of one horizontal layer. As in the proof of the previous theorem, after the chip removal the new jumpers share the leftmost (as in Fig. 9) or the rightmost (as in the figure below) vertex of the chip, depending on the parity of the number of the chip.



Let us number the vertices of the graph in such a way that for vertices on the same horizontal layer the directions of numbering interchange: two layers are numbered from left to right, then two layers are numbered from right to left, etc. (see Fig. 9). Consider the operation of removing the chip H_k. We may assume that the numbering starts from 1. Let the weights of the jumpers in the previous chip be equal to $w_{1,k+1}, w_{2,k+1}, \dots, w_{k-1,k+1}$, and the orientation of the jumpers be "hidden" in the signs of the weights. For uniformity, we assume that the edge $k \to (k+1)$ is present and $w_{1,k+1} = 0$. The edges of the initial graph have weight 1. Let us calculate the weights of the new jumpers $W_{2k+2+i,3k+4}$.

First, calculate the Pfaffian of H_k. The matchings of H_k are uniquely determined by the edge $j \to (k+1)$:

$$\mathrm{Pf}(H_k) = \sum_{i=1}^{k-1} \mathrm{sgn}(1, k+2, 2, k+3, \ldots, i-1, k+i, \underline{i, k+1}, i+1, k+i+1, \ldots, k, 2k, 2k+1, 2k+2)$$
$$\times w_{1,k+2} \cdot \ldots \cdot w_{i-1,k+i} \cdot w_{i,k+1} \cdot w_{i+1,k+i+1} \cdot \ldots \cdot w_{2k+1,2k+2}$$
$$= \sum_{i=1}^{k-1} (-1)^{\frac{k(k-1)}{2} - i} \cdot w_{i,k+1}.$$

Second, calculate the Pfaffian of the chip with the removed external vertices of the edges $(2k+2) \to (3k+4)$ and $(2k+2+i) \to (2k+1-i)$. For $i < k$, a matching in the graph $H_k \setminus \{2k+1-i, 2k+2\}$ is uniquely determined by the edge $j \to (k+1)$, where $i \le j \le k$; and for $i = k$, there exists only one matching. Analogously to (11), (12), we obtain for $i < k$,

$$\mathrm{Pf}(H_k \setminus \{2k+1-i, 2k+2\}) \sum_{j=i}^{k-1} (-1)^{\frac{k(k-1)}{2} - (j-1)} \cdot w_{j,k+1};$$

and for $i = k$,

$$\mathrm{Pf}(H_k \setminus \{k+1, 2k+2\}) = (-1)^{\frac{k(k-1)}{2}}.$$

Now use formula (4) to find the weight $W_{2a+2+i,3a+3}$ which corresponds to the contact edges $(2k+1-i) \to (2k+2+i)$ and $(2k+2) \to (3k+4)$. For $i < k$,

$$W_{2k+2+i,3k+4}$$
$$= \delta_{i1} + (-1)^{(2k+1-i)+(2k+2)} \cdot w_{2k+1-i,2k+2+i} \cdot w_{2k+2,3k+4} \cdot \frac{\mathrm{Pf}(H_k \setminus \{2k+1-i, 2k+2\})}{\mathrm{Pf}(H_k)}$$
$$= \delta_{i1} + (-1)^{i+1} \cdot \frac{\sum\limits_{j=i}^{k-1} (-1)^{j+1} w_{j,k+1}}{\sum\limits_{j=1}^{k-1} (-1)^{j+1} w_{j,k+1}}.$$

Similarly, for $i = k$ we have

$$W_{3k+2,3k+4} = (-1)^{k+1} \frac{1}{\sum\limits_{j=1}^{k-1} (-1)^{j+1} \cdot w_{j,k+1}}.$$

It remains to calculate the Pfaffian of the graph obtained after the removal of the chip H_n. As in the previous theorem, it is given by formula (13) (see. Fig. 7):

$$\mathrm{Pf}(H_{n+1}) = (-1)^{\frac{k(k-1)}{2}} \sum_{i=1}^{k} (-1)^{i+1} \cdot w_{i,k+1}.$$

Let us change the notation: let $A_{k-1,j} = (-1)^{j+1} w_{j,k+1}$, $j = 1, 2, \ldots, k$. In particular, $A_{0,1} = 1 = \delta_{11}$ (these equalities mean that the chip H_1 has no jumpers). In this notation,

$W_{2k+2+i,3k+4} = (-1)^{i+1} A_{k,i}$, and we can rewrite the obtained recurrence in the form

$$A_{k,i} = \delta_{i1} + \frac{\sum\limits_{j=i}^{k-1} A_{k-1,j}}{\sum\limits_{j=1}^{k-1} A_{k-1,j}}, \quad i < k; \qquad A_{k,k} = \frac{1}{\sum\limits_{j=1}^{k-1} A_{k-1,j}}.$$

It is not hard to check that the following weights solve this recurrence:

$$A_{k-1,i} = \frac{t_{k,k-i}}{t_{k,k}} \quad \text{for } i > 1, \qquad A_{k-1,1} = 2 = \frac{2t_{k,k-1}}{t_{k,k}} = \frac{t_{k,k-1} + t_{k,k}}{t_{k,k}}.$$

Therefore, $\operatorname{Pf}(H_k) = \frac{t_{k+1,k+1}}{t_{k,k}}$ for $n \geq k > 1$,

$$\operatorname{Pf}(H_1) = 1 = \frac{t_{2,2}}{t_{1,1}} \operatorname{Pf}(H_{n+1}) = (-1)^{\frac{k(k-1)}{2}} \frac{t_{n+2,n+2}}{t_{n+1,n+1}}.$$

Then the matching number of Arnold's snake of order n equals

$$\begin{aligned} \left|\operatorname{Pf}(H_1) \cdot \operatorname{Pf}(H_2) \cdot \ldots \cdot \operatorname{Pf}(H_n) \cdot \operatorname{Pf}(H_{n+1})\right| &= \frac{t_{2,2}}{t_{1,1}} \cdot \frac{t_{3,3}}{t_{2,2}} \cdot \ldots \cdot \frac{t_{n_1,n+1}}{t_{n,n}} \cdot \frac{t_{n+2,n+2}}{t_{n+1,n+1}} \\ &= \frac{t_{n+2,n+2}}{t_{1,1}} = t_{n+2,n+2}. \end{aligned}$$

□

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