Solutions to the First Inner Olympiad 2019-2020

1. Consider as an example $f(x) = x^2 \sin x$.

2. Making a substitution $y = x - 1004$ we get $\int\limits_{-1004}^{1004} (y+1004)(y+1000)\dots(y+4)y(y+4)\dots(y-1000)(y-1004)\,\mathrm{d}y = \int\limits_{-1004}^{1004} (y^2-1004^2)(y^2-1000^2)\dots(y^2-4^2)y\,\mathrm{d}y$. The function inside the integral is odd and, thus, its integral is 0.

3. Using the simple inequality $(f'(x))^2 \geq 2f'(x) - 1$ we get $\int\limits_0^1 (f'(x))^2\,\mathrm{d}x \geq \int\limits_0^1 (2f'(x)-1)\,\mathrm{d}x = 1$.

4. Let $z(x) = \int\limits_{\pi}^{x} \frac{\mathrm{d}t}{y(t)}$. Then, $z'(x) = \frac{1}{y(t)}$, $z(\pi) = 0$, $z'(\pi) = 1$. $z''(x) = -\frac{y'(x)}{y(x)^2}$. Thus, the original equality from the statement equivalent to $z'' + z = -1$ with $z(\pi) = 0$ and $z'(\pi) = 1$. The general solution is $c_1 \cos x + c_2 \sin x$ and the particular solution is $z(x) = -1$. By applying boundary constraints we get $z(x) = -1 - \cos x - \sin x$. Thus, $y(x) = \frac{1}{z'(x)} = \frac{1}{\sin x - \cos x}$.

5. At first, factorize $2^{13} - 2 = 2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$. For all these primes $p - 1 | 12$, thus by Little Fermat's Theorem $p | x^{12} - 1$. Hence, gcd is $2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$.

6. $(\frac{a_1}{a_1+b_1} \cdots \frac{a_n}{a_n+b_n})^{\frac{1}{n}} + (\frac{b_1}{a_1+b_1} \cdots \frac{b_n}{a_n+b_n})^{\frac{1}{n}} \leq \frac{\frac{a_1}{a_1+b_1} + \ldots + \frac{a_n}{a_n+b_n}}{n} + \frac{\frac{b_1}{a_1+b_1} + \ldots + \frac{b_n}{a_n+b_n}}{n} = 1.$

7. $\det(A^2 - I_n) = \det(A - I_n)(A + I_n) = \det(A - A^T A)(A + A^T A) = \det^2 A \det(I_n - A)^T (I_n + A)^T = \det^2 A \det(I_n - A^2) = \det^2 A (-1)^n \det(A^2 - I_n)$.

8. Let $b_{i,j} = a_{i,j} + 1$. Then $b_{i,j+1} = b_{i,j}^2$ and $b_{i,n} = (b_{i,0})^{2^n}$. $\lim\limits_{n\to\infty} a_{n,n} = \lim\limits_{n\to\infty} b_{n,n} - 1 = \lim\limits_{n\to\infty} \left(\frac{x}{2^n} + 1\right)^{2^n} - 1 = e^x - 1$.

9. Since the roots of the polynomial are reals then the roots of all the derivatives are reals. Consider the derivative $n-2$: $P^{(n-2)}(x) = \frac{n!}{2}x^2 + (n-2)!a_{n-2}$. For this polynomial to have real roots we need $a_{n-2} \leq 0$.

10. $f \equiv 1$ is the solution with $f(f([0,1])) = \{1\}$.

All the functions that satisfy $f \equiv 0$ on $[0, \max\limits_{y\in[0,1]} f(y)]$ are also solutions with $f(f([0,1])) = \{0\}$.

Now, we will show that $\{0\}$ and $\{1\}$ are the only answers.

Suppose there exists $c \in (0,1]$ such that $f(c) = c$. By setting x to c we get: $\int\limits_0^c f(y)\,\mathrm{d}y = \int\limits_0^{f(c)} f(y)\,\mathrm{d}y = f(f(c)) = c$. Thus, $f \equiv 1$ on $[0,c]$. Since, $f(c) = c$ we get that $c = 1$ and $f \equiv 1$.

Consider the opposite case. $f(0) = 0$, otherwise, f intersects $y = x$ ($f(1) \leq 1$) and c exists. Since f is continuous, we get $f([0,1]) = [0,d]$. If $d = 0$ then $f \equiv 0$ and we are done. By the statement we get that for all $z \in [0,d]$ $f(z) = \int\limits_0^z f(y)\,\mathrm{d}y$, because for each z there exists $f(x) = z$. By differentiating both sides we get $f'(z) = f(z)$ on $[0,d]$. Solving we get $f(z) = Ae^z$ on $[0,d]$. However, $f(0) = 0$ and, thus, $A = 0$ and $f \equiv 0$ on $[0,d]$. This means that $f(f([0,1])) = f([0,d]) = \{0\}$.