1. Substitute $g(x) = \frac{f(x)}{x}$, we get $g(xy) = g(x) + g(y) - 2$.
Substitute $h(x) = g(x) - 2$, we get $h(xy) = h(x) + h(y)$.
Substitute $k(x) = h(e^x)$, we get $k(x + y) = k(x) + k(y)$ and $k(x) = cx$.

2. At first, we prove that for $x \geq 0$ $f(x) > 0$. If $x \geq 0$ then $x = \sqrt{2}t$ and $f(x) = f(t)^2 \geq 0$. Let t be the smallest value for which $f(t) = 0$. But for $s = \frac{t}{\sqrt{2}}$ $f^2(s) = f(t) = 0$. Thus, t can be as close to zero as possible, and for $s \geq t$ $f(s) = f(\sqrt{x^2 + t^2}) = 0$. Meaning that f is 0. Hence, we conclude that $f : [0, \infty) \to \mathbb{R}^+$.

Now, we can take the logarithm of both parts. $\ln f(\sqrt{x^2}) + \ln f(\sqrt{y^2}) = \ln f(\sqrt{x^2 + y^2})$. Here we take $g(t) = \ln f(\sqrt{t})$ and $g(x^2) + g(y^2) = g(x^2 + y^2)$. Which is equivalent to Cauchy equation with the solution $g(t) = kt$.

3. How to get to the solution:
$f(x+1) = f(x)+f(1)+x^2$. Simplifying, $f(x+1) = (x+1)f(1)+\frac{x(x-1)(2x-1)}{6} = (x+1)f(1)+\frac{x^3}{3}-\frac{x^2}{2}+\frac{x}{6}$.
Consider now $f(x) = g(x) + \frac{x^3}{3} - \frac{x^2}{2}$.
$g(x + y) + \frac{(x+y)^3}{3} - \frac{(x+y)^2}{2} = g(x) + \frac{x^3}{3} - \frac{x^2}{2} + g(y) + \frac{y^3}{3} - \frac{y^2}{2} + xy(x + y - 1)$.
We get $g(x+y) = g(x)+g(y)$ and since g is continuous $g(x) = cx$ for some c. Hence, all solutions are $f(x) = \frac{x^3}{3} - \frac{x^2}{2} + cx$.

4. Since $g \circ f$ is injective then f and g are injective. $f(x^{2019}) = f(g(f(x))) = f(x)^{2018}$. Values $x \in \{-1, 0, 1\}$ satisfy $f(x) = f(x^{2019}) = f(x)^{2018}$. Thus, for $x \in \{-1, 0, 1\}$ $f(x) \in \{0, 1\}$ and f is not injective. Contradiction.

5. Let $s = \max\limits_{x\in[0,1]}$ and a such that $|f(a)| = s$. Then, $|f(a)| = \frac{1}{3}\left|f\left(\frac{a+1}{2}\right) + f\left(\frac{a}{2}\right)\right| \leq \frac{2}{3}|f(a)|$. Hence, $|f(a)| = 0$ and $f \equiv 0$.

6. Note that if $xP(x)$ satisfies the condition, then $P(x)$ also satisfies the condition.
Suppose, that $x \neq |P(x)$ and $P(x)$ is not a constant.
There is always exists a root that is not in $\mathbb{R}^+$. Because we can take a root $a \in \mathbb{R}^+$, there exists $\alpha \neq\in \mathbb{R}^+$ such that $\alpha^2 = a$. Note that α is also a root.
Suppose that $\{z|P(z)$ and $z \neq\in \mathbb{R}\}$ is not empty. Then pick z_0 with the smallest argument. However, $\exists z_1$ such that $z_1^2 = z_0$ and $\arg z_1 = \frac{\arg z_0}{2}$ which is the root. Contradiction.
Thus, the solutions are $P(x) = x^k$, $P(x) = 0$ and $P(x) = 1$.

7. If $f(s) = f(t) = p$ then $sp^5 = tp^5 \Rightarrow s = t$, hence f is bijective. Choose a with $f(a) = I$ and take $t = a$ then $sf(s)^3 = af(s)^2 \Rightarrow sf(s) = a$, hence $xf(x) = a$ for all x. Then $sf(s)f(s)^2f(t)^2 = tf(t)f(t)^2f(s)^2$ reduces to $f^2(s)f^2(t) = f^2(t)f^2(s)$ for all s, t. Since f is bijective this means $x^2y^2 = y^2x^2$ for all $x, y \in S_n$.
If $n \geq 4$ this is simply not true.
If $n = 2$ then any bijection satisfies. (2 solutions)
If $n = 3$ all functions $f(x) = x^{-1}a$ where $a \in S_3$. (6 solutions)

8. *First solution.* First we show that f is infinitely times differentiable. By substituting $a = \frac{1}{2}t$ and $b = 2t$ we get $f'(t) = \frac{f(2t)-f(\frac{1}{2}t)}{\frac{3}{2}t}$. If the right side is k times differentiable, then f' is also and, thus, f is $k + 1$ times differentiable.
Substitute $b = e^ht$ and $a = e^{-h}t$: $f(e^ht) - f(e^{-h}t) - (e^ht - e^{-h}t)f'(t) = 0$. Then we differentiate three times by h.
We get: $e^{3h}t^3f'''(e^ht)+3e^{2h}t^2f''(e^ht)+e^htf'(e^ht)+e^{-3h}t^3f'''(e^{-h}t)+3e^{-2h}t^2f''(e^{-h}t)+e^{-h}tf'(e^{-h}t) - (e^ht + e^{-h}t)f'(t) = 0$. Then, we take limit $h \to 0$: $2t^3f'''(t) + 6t^2f''(t) = tf'''(t) + 3f''(t) = (tf(t))''' = 0$. Thus, $f(t) = C_1t + \frac{C_2}{t} + C_3$.
Second solution. f is infinitely differentiable, then by Taylor with $b \to a$:

$$f(b) - f(\sqrt{ab}) - f'(\sqrt{ab})(b - \sqrt{ab}) = \frac{1}{2}f''(\sqrt{ab})(b - \sqrt{ab})^2 + \frac{1}{6}f'''(\sqrt{ab})(b - \sqrt{ab})^3 + o((\sqrt{b} - \sqrt{a})^3)$$

and

$$f(a) - f(\sqrt{ab}) - f'(\sqrt{ab})(a - \sqrt{ab}) = \frac{1}{2}f''(\sqrt{ab})(a - \sqrt{ab})^2 + \frac{1}{6}f'''(\sqrt{ab})(b - \sqrt{ab})^3 + o((\sqrt{b} - \sqrt{a})^3)$$

Subtracting we get: $0 = \frac{1}{2}f''(\sqrt{ab})(\sqrt{b}-\sqrt{a})^3(\sqrt{b}+\sqrt{a}) + \frac{1}{6}f'''(\sqrt{ab})(\sqrt{b}-\sqrt{a})^3(b^{3/2}+a^{3/2}) + o((\sqrt{b}-\sqrt{a})^3)$. By setting $b \to a$ $0 = \frac{1}{2}f''(a) + \frac{1}{6}f'''(a)a$.

9. $g(x) = x^{2015} + x$. $f = g^{-1} \circ \int_0^x f(t)\,\mathrm{d}t$. g^{-1} is continuous and differentiable and $\int_0^x f(t)\,\mathrm{d}t$ is continuous and differentiable, then f is also continuous and differentiable.

Differentiating: $f = 2015 f^{2014} f' + f'$. Multiplying by $2f'$ we can see that $2ff' \geq 0$, i.e., f^2 is increasing. Since, f is continuous and, suppose, $f(x) \geq 0$, f has to be increasing.

$f(1)^{2015} + f(1) = \int_0^1 f(x)\,\mathrm{d}x \leq f(1)$. Therefore, $f \equiv 0$.

The case with $f(x) \leq 0$ is analogous.

10. Note that we can work only with $x \geq 0$. Let $a(x) = x^2 + c$.

Suppose that $c > \frac{1}{4}$. Then $a(x) > x$ for all $x \geq 0$. Let $a_n = a^n(0)$: $a_0 = 0$, $a_1 = c$ and so on. If $I_n = [a_n, a_{n+1})$ then $f : I_n \to I_{n+1}$ is bijective. Hence for $x \geq 0$ there exists unique $k \geq 0$ and $y \in [0, c)$ such that $x = a^k(y)$. Now, take any continuous $\phi : [0, c] \to \mathbb{R}$ with $\phi(0) = \phi(c)$ and extend to $[0, \infty)$ with the rule $f(x) = f(a^k(y)) = \phi(y)$. Since f is even we simply extend to $\mathbb{R}$.

Suppose that $c < \frac{1}{4}$. Then $a(x)$ has two fixed points $p < q$. If $b \in [0, q)$, $b_n = a^n(b) \to p$ then $f(b) = f(p)$. If $b \in [q, \infty)$, $b_n = a^{-n}(b) \to q$ then $f(b) = f(q)$. Since f is continuous, f is constant.

The case $c = \frac{1}{4}$ is similar to the previous one.