Number theory problems

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Numbers a, b, n, k, d below are positive integers, p is prime number.

1. $a^2 + b \vdots b^2 + a$ and $a + b > 2$. Prove that $b^2 + a$ is composite integer.
2. Solve equation: $a(a + 1) = b(b + 2)$.
3. Prove that there exists 2020 consecutive positive integers such that exactly 19 integers among them are prime numbers.
4. Prove that $239^{30} + 30^{239}$ is composite integer.
5. $p = 4k + 3$, $a^2 + b^2 \vdots p$. Prove that $b \vdots p$ (Note: use quadratic characters)
6. $p > 2$, $2^p - 1 \vdots d$. Prove that $d = 2kp + 1$.
7. $p = 3k + 2$. Prove that for any a equation $x^3 \equiv a \pmod p$ has exactly one solution (Note: use generator modulo p).
8. Let $f(n, k) = \#\{d = k \dots n \mid n \vdots d\}$. Find $f(1001, 1) + f(1002, 2) + \dots + f(2000, 1000)$.
9. Let $a_1, \dots, a_{10}$ be distinct positive integers. Let $M = \{a_1, \dots, a_{10}, -a_1, \dots, -a_{10}\}$. Prove that there exists nonempty $S \subset M$ such that $\forall i : \{a_i, -a_i\} \not\subset S$ and $\sum\limits_{x \in S} x \vdots 1001$.
10. Positive integer n is called Carmichael (or Fermat pseudoprime) number if $\forall a : a^n - a \vdots n$. Prove that n is Carmichael number iff for all prime divisor p of n: $p^2 \nmid n$ and $p - 1 | n - 1$.
11. Prove that for any $n > 1$ number $3^n - 1$ is not divisible by $2^n - 1$.
12. (Kummer's lemma). Given a, b and p. Let k_1 be maximum d such that $C^a_{a+b} \vdots p^d$. Let k_2 be number of carryings in process of addition in columns of numbers a and b in numeral system with base p. Prove that $k_1 = k_2$.
13. Someone calculated pairwise gcd of 10 positive integers. Is it possible that 45 resulting values equal to $1, 2, 3, \dots, 45$?
14. You are given a multiset S of 101 integers. It is known that $\forall x \in S$: $\exists S_1, S_2 \subset S$: $|S_1| = |S_2| = 50$, $S_1 \cap S_2 = \varnothing$, $S_1 \cup S_2 \cup \{x\} = S$ and $\sum\limits_{y \in S_1} y = \sum\limits_{z \in S_2} z$. Prove that all numbers in S are equal. (Bonus: try to prove it if S consists of real numbers, not integers)
15. Find all n such that $n^2 + 3 \vdots \phi(n)$.
16. Does there exist a field such that its multiplicative group is isomorphic to its additive group?