Theorem (Cayley-Hamilton)**.** *If A is $n \times n$ matrix and I_n is the $n \times n$ identity matrix, then the* characteristic *polynomial of A is defined as $p(\lambda) = \det(\lambda I_n - A)$. For example, for $n = 2$ $p(\lambda) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A)$.*

Replacement of the scalar eigenvalues λ with the matrix A in $p(\lambda)$ gives 0, i.e., $p(A) = 0$,

1. If M is 3×3 matrix with $M^T M = I$ and $\det M = 1$. Find $\det(M - I)$.
2. Let S be the subspace of $M_{n\times n}$ (the vector space of all real $n \times n$ matrices) generated by all matrices of the form $AB - BA$ with $A, B \in M_{n\times n}$. Show that $\dim(S) = n^2 - 1$.
3. Let $n \geq 3$. Let A be $n \times n$ matrix such that $a_{ij} \in \{-1, 1\}$ for all $1 \leq i, j \leq n$. Suppose that $a_{k1} = 1$ for all $1 \leq k \leq n$ and $\sum_{k=1}^{n} a_{ki}a_{kj} = 0$ for all $i \neq j$. Show that n is multiple of 4.

Definition. *Two norms $||\cdot||_1$ and $||\cdot||_2$ are equivalent on linear space X if there exists C and D such that for every $x \in X$: $C||x||_1 \leq ||x||_2 \leq D||x||_1$.*

4. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Consider two norms: $||A||_E = (a^2 + b^2 + c^2 + d^2)^{\frac{1}{2}}$ and $||A||_{op} = \sup_{||v||_2=1} ||Av||$. Prove that they are equivalent.
5. Give an example of two 2×2 matrices such that the norm (*op* norm from Problem 4) of the product is less than the product of the norms.
6. Let $A, B \in M_n(\mathbb{C})$ such that $A^2 = A$, $B^2 = B$ and $A - B$ is invertible. Prove that $AB - I$ is invertible.
7. Let A, B, C, D be complex $n \times n$ matrices with A and C invertible. If $A^k B = C^k D$ for all $n \in \mathbb{N}$ then $B = D$.
8. Let $n \in \mathbb{N}$, $n \geq 3$. Find $X \in \mathcal{M}_2(\mathbb{R})$ such that $X^n + X^{n-2} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.
9. Consider the matrix $A = \begin{pmatrix} 2 & -2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix}$. Let $B \in M_{3,2}(\mathbb{R})$ and $C \in M_{2,3}(\mathbb{R})$ so that $B \cdot C = A$. Find $\det(CB)$ and $\operatorname{tr}(CB)$.

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