1. $\det(M-I)=\det(M^TM-M^T)=\det(I-M^T)=-\det(M^T-I)=-\det(M-I)$. Thus, $\det(M-I)=0$.

2. Let V be the subspace of matrices with trace 0, obviously, its dimension is not bigger than n^2-1 since m_{nn} is fixed.

NB. It is well-known that subspace generated by $AB-BA$ also equals V.

Let us show directly that each matrix of trace 0 is a commutator. For $p,q\in[1,n]$ let M_{pq} be the matrix with one in position (p,q) and zero elsewhere. Now take $A=M_{pq}, B=M_{rs}$ and consider $AB-BA$.

$AB_{ij}=0$ unless $i=p, q=r, s=j$ in which case we have $AB_{ps}=1$.

$BA_{ij}=0$ unless $i=r, s=p, q=j$ in which case we have $AB_{rq}=1$.

If $\alpha\neq\beta$ we have $M_{\alpha\beta}=AB-BA$ with $A=M_{\alpha\alpha}$ and $B=M_{\alpha\beta}$.

If $\alpha<n$ we have $M_{\alpha\alpha}-M_{nn}=AB-BA$ with $A=M_{\alpha n}$ and $B=M_{n\alpha}$.

3. At first, $0=\sum\limits_{k=1}^{n}a_{k1}a_{k2}=\sum\limits_{k=1}^{n}a_{k2}$. Thus, n has to be divisible 2. Suppose that $n=2m$.

From $\sum\limits_{k=1}^{n}a_{kj}=0$ we know that the half of values are -1, because of that $\prod\limits_{k=1}^{n}a_{kj}=(-1)^m$.

Analogously, from $\sum\limits_{k=1}^{n}a_{k2}a_{k3}=0$ we get $(-1)^m=\prod\limits_{k=1}^{n}a_{k2}a_{k3}=\prod\limits_{k=1}^{n}a_{k2}\prod\limits_{k=1}^{n}a_{k3}=(-1)^m(-1)^m$. Thus, m is divisible by 2.

4. Consider vector $u=(x,y)$ with $x^2+y^2=1$. $||Au||^2=(ax+by)^2+(cx+dy)^2\leq(a^2+b^2)(x^2+y^2)+(c^2+d^2)(x^2+y^2)=||A||_E^2$. Thus, $\beta=1$.

By the definition, $||A||_{op}^2\geq a^2+c^2$ (take $u=(1,0)$) and $||A||_{op}^2\geq b^2+d^2$ (take $u=(0,1)$). Thus, $||A||_{op}^2\geq\frac{1}{2}(a^2+b^2+c^2+d^2)=\frac{1}{2}||A||_E^2$, and, hence, $\alpha=\frac{1}{\sqrt{2}}$

5. $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ and $B=\begin{pmatrix}0&0\\0&1\end{pmatrix}$. $|A|=|B|=1$ and $|AB|=0$.

6. Let v be a non-zero vector with $(AB-I)v=0$. $ABv=A^2Bv=Av$. Then $Bv-v\in\ker A$. Since $B^2=B$, we also have $Bv-v\in\ker B$. However $Bv-v\in\ker B\cap\ker A=\{0\}$, because $A-B$ is invertible. Thus, $Bv=v$ and, similarly $Av=v$. By that $(A-B)v=0$. Contradiction, since $A-B$ is invertible.

7. Let $f(x)$ and $g(x)$ be characteristic polynomals of A and C. Consider $h(x)=f(x)g(x)-f(0)g(0)$. By the condition of the problem, $h(A)B=h(C)D$. Then, using the fact that $f(A)=0$, $g(C)=0$ and, consequently, $f(A)g(A)B=f(C)g(C)D=0$, we get $f(0)g(0)B=f(0)g(0)D$. Since, $f(0)=\det A$, $g(0)=\det C$ and A and C are invertible, then $f(0)g(0)\neq0$.

8. $0=\det(X^n+X^{n-2})=\det X^{n-2}\det(X^2+I_2)$.

Let's suppose that $\det(X^2+I_2)=0$. Then $0=\det(X^2+I_2)=\det(X+iI_2)\det(X-iI_2)=\det(X+iI_2)\overline{\det(X+iI_2)}$. Thus, $\det(X+iI_2)=0$.

Using the identity $\det(A+xB)=\det Bx^2+(\mathrm{tr}A\cdot\mathrm{tr}B-\mathrm{tr}(AB))x+\det A$ for $A=X$, $B=I_2$ and $x=i$, we get $0=\det(X+iI_2)=-1+i\mathrm{tr}X+\det X$. So, $\det X=1$ and $\mathrm{tr}X=0$. Which by Cayley-Hamilton $0=X^2-\mathrm{tr}XX+\det X=X^2+I_2$. But then $X^n+X^{n-2}=0$.

Hence $\det X=0$. Suppose that $a=\mathrm{tr}X$. Again, by Cayley-Hamilton we get $X^2-aX=0$. Substituting in $X^n+X^{n-2}=(a^{n-1}+a^{n-3})X$. By taking trace, we obtain $2=(a^{n-1}+a^{n-3})\mathrm{tr}X=a^n+a^{n-2}$. Thus, $a=1$ ($a=(-1)^{n-1}$) and $X^2-X=0$ ($X^2-(-1)^{n-1}X=0$). Replacing X^n by X ($(-1)^{n-1}X$) and X^{n-2} by X ($(-1)^{n-3}$), we get $2X=\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$ ($(-1)^n2X=\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$).

9. Since $CB\in M_2(\mathbb{R})$ from Cayley-Hamilton we get that $(CB)^2-\mathrm{tr}(CB)\cdot CB+\det(CB)\cdot I_2=O_2$. After multiplying by B to the left and by C to the right we get $O_3=(BC)^3-\mathrm{tr}(CB)\cdot(BC)^2+\det(CB)\cdot(BC)=A^3-\mathrm{tr}(CB)A^2+\det(CB)A$. Again, from Cayley-Hamilton we know that $A^3-5A^2+6A=O_3$. Thus, $\mathrm{tr}(CB)=5$ and $\det(CB)=6$.