Theorem (Jensen's Inequality). *For a convex function f holds $f(tx_1+(1-t)x_2) \leq tf(x_1)+(1-t)f(x_2)$ for all $t \in [0,1]$.*

Theorem (Cauchy-Schwarz's Inequality). *If $f, g : [a,b] \to \mathbb{R}$ are integrable, then*

$$\left| \int_a^b f(x)g(x)\,dx \right|^2 \leq \left(\int_a^b f^2(x)\,dx \right) \cdot \left(\int_a^b g^2(x)\,dx \right),$$

with equality when $|g(x)| = c \cdot |f(x)|$

1. Prove that $\int_0^1 \frac{\mathrm{d}x}{\sqrt{x+2^{-x}}} < 0.8$.

2. Prove that the following inequality holds for all $x > 1$, $\int_1^x \frac{\sqrt{t^2+1}}{t}\,\mathrm{d}t > \sqrt{\ln^2 x + (x-1)^2}$.

3. Consider $a \in \mathbb{R}_+$ and the convex function $f : [0,a] \to \mathbb{R}$ with $f(0) = 0$. Prove that $\int_0^a f(x)\,\mathrm{d}x \geq \frac{a^2}{c^2} \int_0^c f(x)\,\mathrm{d}x$ for all $c \in (0,a)$.

4. If $f : [0,\infty) \to \mathbb{R}$ is a convex function and $a, b, c > 0$, show that: $\int_0^a f(x)\,\mathrm{d}x + \int_0^b f(x)\,\mathrm{d}x + \int_0^c f(x)\,\mathrm{d}x + \int_0^{a+b+c} f(x)\,\mathrm{d}x \geq \int_0^{a+b} f(x)\,\mathrm{d}x + \int_0^{a+c} f(x)\,\mathrm{d}x + \int_0^{b+c} f(x)\,\mathrm{d}x$.

5. Let $f : [0,1] \to \mathbb{R}$ be concave and $f(0) = 1$. Prove that $\frac{3}{2} \int_0^1 xf(x)\,\mathrm{d}x \leq \int_0^1 f(x)\,\mathrm{d}x - \frac{1}{4}$.

6. Prove that if $f : [0,\infty) \to \mathbb{R}$ is continuous and satisfies the inequality $2015 \int_0^x f^2(t)\,\mathrm{d}t \leq \left(\int_0^x f(t)\,\mathrm{d}t \right)^2$, then $f(x) \equiv 0$.

7. Let $f_1, f_2, \ldots, f_n : [0,1] \to (0,\infty)$ be continuous functions and let σ be a permutation of the set $\{1, 2, \ldots, n\}$. Prove that $\prod_{i=1}^n \int_0^1 \frac{f_i^2(x)}{f_{\sigma(i)}(x)}\,\mathrm{d}x \geq \prod_{i=1}^n \int_0^1 f_i(x)\,\mathrm{d}x$.

8. Let $f : [0,1] \to \mathbb{R}$ be a function with a continuous derivative f'. Prove that if $f\left(\frac{1}{2}\right) = 0$, then $\int_0^1 (f'(x))^2\,\mathrm{d}x \geq 12 \left(\int_0^1 f(x)\,\mathrm{d}x \right)^2$.

9. Let $f \in C^1[0,1]$ with $\int_0^1 f(t)\,\mathrm{d}t = 0$. Prove that $\int_0^1 f(t)^2\,\mathrm{d}t \leq \frac{1}{2} \int_0^1 f'(t)^2\,\mathrm{d}t$.

10. Let $F = \{f : [0,1] \to [0,\infty) \big| f \text{ - continuous}\}$ and let $n \geq 2$ be a positive integer. Determine the least real constant c such that $\int_0^1 f(\sqrt[n]{x})\,\mathrm{d}x \leq c \int_0^1 f(x)\,\mathrm{d}x$ for every $f \in F$.