1. Note that $2^{-x} \geq 1 - \frac{1}{2}\sqrt{x}$ for $0 \leq x \leq 1$.
This gives us, $\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}+2^{-x}} \leq \int_0^1 \frac{\mathrm{d}x}{1+\frac{1}{2}\sqrt{x}} \approx 0.756$.

2. Consider the function $f(t) = \ln t$. Consider two points $A(1,0)$ and $B(x, \ln x)$. Then $\sqrt{\ln^2 x + (x-1)^2}$ is just the length of AB. Which is smaller than the length of the arc AB of f: $\int\limits_1^t \sqrt{1+(f'(t))^2}\,\mathrm{d}t = \int\limits_1^x \sqrt{1+\frac{1}{t^2}}\,\mathrm{d}t = \int\limits_1^x \frac{\sqrt{1+t^2}}{t}$.

3. Let $\lambda = \frac{c}{a}$. As $f(0) = 0$ by convexity we get $f(\lambda x) = f(\lambda x + (1-\lambda)0) \leq \lambda f(x)$.

$$\int_0^c f(x)\,\mathrm{d}x = \int\limits_0^{\lambda a} f(x)\,\mathrm{d}x = \lambda \int\limits_0^{a} f(\lambda x)\,\mathrm{d}x \leq \lambda^2 \int\limits_0^{a} f(x)\,\mathrm{d}x.$$

4. This is equivalent to

$$\int\limits_0^a f(x)\,\mathrm{d}x + \int\limits_{b+c}^{a+b+c} f(x)\,\mathrm{d}x \geq \int\limits_b^{a+b} f(x)\,\mathrm{d}x + \int\limits_c^{a+c} f(x)\,\mathrm{d}x \Leftrightarrow \int\limits_0^a f(x) + f(x+b+c) - f(x+b) - f(x+c)\,\mathrm{d}x \geq 0.$$

By the convexity we get $f(x+b) \leq \frac{c}{b+c} f(x) + \frac{b}{b+c} f(x+b+c)$ and $f(x+c) \leq \frac{b}{b+c} f(x) + \frac{c}{b+c} f(x+b+c)$, by adding we obtain $f(x+b) + f(x+c) \leq f(x) + f(x+b+c)$.

5. Let $g(x) = f(x) - 1$. So, $g(x)$ is concave and $g(0) = 0$. Our problem becomes $\int\limits_0^1 g(x)(2-3x) \geq 0$.
Let $A = g(\frac{2}{3})$. Consider the secant line from $(0,0)$ to $(\frac{2}{3}, A)$. Since g is concave we get $g(x) \geq \frac{3}{2}Ax$.
Consider the secant line from $(0,0)$ to $(x, g(x))$, $A = g(\frac{2}{3}) \geq \frac{1}{x} g(x) \frac{2}{3} \Rightarrow g(x) \leq \frac{3}{2}Ax$.
As a result, $g(x)(2-3x) \geq \frac{3}{2}Ax(2-3x)$ for all x, but $\int\limits_0^1 x(2-3x)\,\mathrm{d}x = 0$.

6. We prove by induction that $f(x) = 0$ for all $x \in [0,k]$.

$$2015 \left(\int\limits_k^{k+1} f^2(x)\,\mathrm{d}x \right) \leq \left(\int\limits_k^{k+1} f(x)\,\mathrm{d}x \right)^2 \leq \left(\int\limits_k^{k+1} \mathrm{d}x \right) \left(\int\limits_k^{k+1} f^2(x)\,\mathrm{d}x \right)$$
$$= \int\limits_k^{k+1} f^2(x)\,\mathrm{d}x.$$

$$\int\limits_k^{k+1} f^2(x)\,\mathrm{d}x = 0 \Rightarrow f(x) = 0 \forall x \in [k, k+1].$$

7. We can use Cauchy-Schwarz inequality $\left(\int\limits_0^1 \frac{f_i^2(x)}{f_{\sigma(i)}(x)} \right) \left(\int\limits_0^1 f_{\sigma(i)}(x) \right)$.

8.

$$\begin{aligned}
\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2 &= \left(\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x + \int_{\frac{1}{2}}^1 f(x)\,\mathrm{d}x\right)^2 \\
&\le 2\left(\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x\right)^2 + 2\left(\int_{\frac{1}{2}}^1 f(x)\,\mathrm{d}x\right)^2 \\
&= 2\left(xf(x)\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} xf'(x)\ \mathrm{d}x\right)^2 + 2\left((1-x)f(x)\Big|_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 (1-x)f'(x)\ \mathrm{d}x\right)^2 \\
&= 2\left(\int_0^{\frac{1}{2}} xf'(x)\ \mathrm{d}x\right)^2 + 2\left(\int_{\frac{1}{2}}^1 (1-x)f'(x)\ \mathrm{d}x\right)^2 \\
&\le 2\int_0^{\frac{1}{2}} x^2\ \mathrm{d}x \int_0^{\frac{1}{2}} \left(f'(x)\right)^2\ \mathrm{d}x + 2\int_{\frac{1}{2}}^1 (1-x)^2\ \mathrm{d}x \int_{\frac{1}{2}}^1 \left(f'(x)\right)^2\ \mathrm{d}x \\
&= \frac{1}{12}\int_0^1 \left(f'(x)\right)^2\ \mathrm{d}x.
\end{aligned}$$

9. Let $g(x) = \int_0^x f(t)\,\mathrm{d}t$. Then $\int_0^1 f(x)^2\,\mathrm{d}x = \int_0^1 f(x)g'(x)\,\mathrm{d}x$, and by integration by parts and $g(0) = g(1) = 0$ we get $\int_0^1 f(x)^2\,\mathrm{d}x = -\int_0^1 f'(x)g(x)\,\mathrm{d}x$.

By Cauchy-Schwarz,

$$\left(\int_0^1 f(x)^2\,\mathrm{d}x\right)^2 = \left(\int_0^1 f'(x)g(x)\,\mathrm{d}x\right)^2 \le \int_0^1 f'(x)^2\,\mathrm{d}x \int_0^1 g(x)^2\,\mathrm{d}x.$$

Again, by Cauchy-Schwarz,

$$g(x)^2 = \left(\int_0^x f(t)\,\mathrm{d}t\right)^2 \le \int_0^x f(t)^2\,\mathrm{d}t \int_0^x \mathrm{d}t \le x\int_0^1 f(t)^2\,\mathrm{d}t.$$

Hence,

$$\int_0^1 g(x)^2\,\mathrm{d}x \le \int_0^1 x\,\mathrm{d}x \int_0^1 f(t)^2\,\mathrm{d}t = \frac{1}{2}\int_0^1 f(t)^2\,\mathrm{d}t.$$

By putting everything together we get what we want.

Theorem (Wirtinger's inequality 1)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function of period 2π, which is continuous, has a continuous derivative and $\int_0^{2\pi} f(x)\,\mathrm{d}x = 0$. Then, $\int_0^{2\pi} f'^2(x)\,\mathrm{d}x \ge \int_0^{2\pi} f^2(x)\,\mathrm{d}x$ with equality for $f(x) = a\sin x + b\cos x$.*

Theorem (Wirtinger's inequality 2)**.** *Let* $f \in C^1[-\infty, \infty]$ *such that* $f(0) = f(a) = 0$. *Then,* $\pi^2 \int\limits_0^a |f(x)|^2 \,\mathrm{d}x \leq a^2 \int\limits_0^a |f'(x)|^2 \,\mathrm{d}x$.

10. Substitute $\sqrt[n]{x} = t$ and get

$$\int_0^1 f(\sqrt[n]{x}) \,\mathrm{d}x = n \int_0^1 t^{n-1} f(t) \,\mathrm{d}t \leq n \int_0^1 f(t) \,\mathrm{d}t,$$

hence $c \leq n$.

For $p > 0$, the function $f_p = x^p$ belongs to F. $\int\limits_0^1 x^{\frac{n}{p}} \leq c \int\limits_0^1 x^p \,\mathrm{d}x$ implies $\frac{n}{n+p} \leq \frac{c}{p+1}$, therefore $c \geq \frac{pn+n}{p+n}$. Finally, $c \geq \lim\limits_{p \to \infty} \frac{pn+n}{p+n} = n$.