Definition. *A real number is* trancendental *if it is not a root of a non-zero polynomial equation with integer coefficients.*

Definition. *A real number x is a* Liouville number *if for every possible integer n, there exist integers p and $q > 1$, such that $0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}$.*

1. Prove that a Liouville number is trancendental.
2. Prove that the number $\sum_{k=1}^{\infty} \frac{1}{(k!)!}$ is trancendental.

Definition (Möbius Function)**.** *The Möbius function $\mu : \mathbb{Z}^+ \to \{-1, 0, 1\}$ is defined as follows:*

$$\mu(x) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is squarefree and } k \text{ is the number of prime divisors} \\ 0 & \text{else} \end{cases}.$$

Theorem (Möbius Inversion Formula)**.** *Suppose that $F, H, f : \mathbb{Z}^+ \to \mathbb{Z}^+$ are functions such that $F(n) = \sum_{d|n} f(d)$ and $H(n) = \prod_{d|n} f(d)$. Then, $f(n) = \sum_{d|n} \mu(d) F(\frac{n}{d})$ and $f(n) = \prod_{d|n} H(\frac{n}{d})^{\mu(d)}$.*

3. Prove Möbius Inversion Formula Theorem.

Definition (Cyclotomic Polynomials)**.** *Let ζ_n be the complex number $e^{\frac{2\pi i}{n}}$. The n^{th} cyclotomic polynomial is*

$$\Phi_n(x) = \prod_{\substack{1 \le k \le n \\ \gcd(k,n)=1}} (x - \zeta_n^k).$$

Properties. *1.* $x^n - 1 = \prod_{d|n} \Phi_d(x)$.

2. $\Phi_n(x) \in \mathbb{Z}[x]$.

3. $\Phi_n(x) = \prod_{d|n} (x^{\frac{n}{d}} - 1)^{\mu(d)}$.

4. Let p be a prime number and n be a positive integer. Then, $\Phi_{pn}(x) = \begin{cases} \Phi_n(x^p), & \text{if } p|n \\ \frac{\Phi_n(x^p)}{\Phi_n(x)} & \text{if } p \neq |n \end{cases}$

5. $\Phi_n(x)$ *is irreducible in* $\mathbb{Z}[x]$.

4. Prove Property 1.
5. a) Suppose that $f(x)$ and $g(x)$ are monic polynomials with rational coefficients. Then, if $f \cdot g \in \mathbb{Z}[x]$ then $f, g \in \mathbb{Z}[x]$.
b) Prove Property 2.
6. Prove Property 3.
7. Prove Property 4.
8. a) Let p be a prime number. Suppose that the polynomial $x^n - 1$ has a double root modulo p, that is, there exists an integer a and a polynomial $f(x) \in \mathbb{Z}[x]$ such that $x^n - 1 \underset{p}{\equiv} (x - a)^2 f(x)$. Then $p|n$.
b) Let n be a positive integer, $d < n$ is a divisor of n and b is any integer. Suppose that p divides $\Phi_n(b)$ and $\Phi_d(b)$, then $p|n$.
9. Let n be a positive integer and x be any integer. Then every prime divisor p of $\Phi_n(x)$ either satisfies $p \underset{n}{\equiv} 1$ or $p|n$.
10. Let a and b be positive integers. Suppose that x is an integer so that $\gcd(\Phi_n(x), \Phi_m(x)) > 1$. Then $\frac{n}{m} = p^k$ for some prime number p and integer k.
11. Let n be a positive integer. Prove that there exist infinitely many prime numbers p with $p \underset{n}{\equiv} 1$.
12. For any positive integer n consider the polynomial $f_n = x^{2n} + x^n + 1$. Prove that for any positive integer m there is a positive integer n such that f_n has exactly m irreducible factors in $\mathbb{Z}[x]$.