1. Compute $\lim\limits_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$.

Solution. Take the natural logarithm. $\lim\limits_{n\to\infty} \ln \frac{\sqrt[n]{n!}}{n} = \lim\limits_{n\to\infty} \frac{\ln\frac{1}{n}+\ln\frac{2}{n}+\ldots+\ln\frac{n}{n}}{n}$. The last sum is the Riemann sum for an integral: $\int\limits_0^1 \ln x\,\mathrm{d}x = (x\ln x - x)|_0^1 = -1$. Thus, the original limit is equal to e^{-1}.

2. Let A and B be two orthogonal $n\times n$ matrices with real entries. What is the maximal possible value of $\det(A+B)$?

Solution. Each column of $A+B$ is a column of A plus a column of B. Columns of A and B are unit vectors, thus, by the triangle inequality, a column of $A+B$ has length at most 2. Since, $\det(A+B)$ is a volume of a parallelepiped built on column vectors, its size does not exceed the product of the length of the vectors, which is 2^n.

If $A = B = E$, we obtain exactly 2^n.

3. Find $\max\limits_{\substack{a,b,c>0\\ a+b+c=1}} a+\sqrt{b}+\sqrt[3]{c}$.

Solution. By straightforward AM-GM, $b+\frac{1}{4}\geq\sqrt{b}$ and $c+\frac{2}{3\sqrt{3}} = c+\frac{1}{3\sqrt{3}}+\frac{1}{3\sqrt{3}}\geq\sqrt[3]{c}$. These equalities hold for $b=\frac{1}{4}$ and $c=\frac{1}{3\sqrt{3}}$.

So, $a+\sqrt{b}+\sqrt[3]{c}\leq a+b+\frac{1}{4}+c+\frac{1}{3\sqrt{3}} = 1+\frac{1}{4}+\frac{1}{3\sqrt{3}}$ when $b=\frac{1}{4}$ and $c=\frac{1}{3\sqrt{3}}$.

4. Find all the roots of $p(x) = (x+1)^{90}+(x-1)^{90}$.

Solution. In order for x to be a root, the numbers $(x+1)^{90}$ and $(x-1)^{90}$ should have the same absolute values but opposite arguments. The only possibility to satisfy that is to have $x = it$.

Let $t = \tan\alpha$. Then $p(x) = (i\tan\alpha+1)^{90}+(i\tan\alpha-1)^{90} = \frac{(i\sin\alpha+\cos\alpha)^{90}+(i\sin\alpha-\cos\alpha)^{90}}{\cos^{90}\alpha} = \frac{e^{i90\alpha}+e^{i90(\pi-\alpha)}}{\cos^{90}\alpha}$. We want $p(x)$ to be zero, thus, $e^{i90\alpha}$ should be equal to $-e^{i90(\pi-\alpha)}$, which is equivalent to $90\alpha = 2\pi k+\pi+90(\pi-\alpha) = (2k+1)\pi-90\alpha$. Thus, $\alpha = \frac{2k+1}{180}\pi$.

To summarize, the roots of $p(x)$ are equal to $i\tan\frac{2k+1}{180}\pi$ for $k\in[0,89]$.

5. A two-dimensional square with side of length 10 is contained in a unit cube of dimension n. What is the least possible n?

Solution. Obviously, the largest distance in a cube, the main diagonal with length $\sqrt{n}$, should exceed the diameter of the square, $10\sqrt{2}$. Thus, $n\geq 200$.

Let us prove that $n = 200$ is enough. The opposite corners of the square should coincide with the opposite corners of the cube, so the square and the cube share the same center O. Thus, it will be enough to find vertices of the cube A and B, such that OA and OB are orthogonal. Then, $A-B--A--B$ is the required square.

$A = (\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$ and $B = (\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2},\ldots,-\frac{1}{2})$ (half of the coordinates are positive) satisfy us.

6. Let $f:\mathbb{R}\to\mathbb{R}$ and $f(1) = 1$. Find all functions f, such that $f(\frac{1}{x}) = \frac{1}{x^2}\cdot f(x)$ for $x\in\mathbb{R}\setminus\{0\}$ and $f(x+y) = f(x)+f(y)$ for $x,y\in\mathbb{R}$.

Solution. Substitute $x = y = 0$ then $f(0+0) = f(0)+f(0)$ and, thus, $f(0) = 0$.

Substitute $x = -y$ then $0 = f(x+(-x)) = f(x)+f(-x)$ and, thus, f is odd.

Assume that $x\neq 0,-1$ and consider $f(\frac{x}{x+1})$.

On one hand, $f(\frac{x}{x+1}) = f(\frac{1}{\frac{x+1}{x}}) = \frac{1}{\left(\frac{x+1}{x}\right)^2}f(\frac{x+1}{x}) = \frac{x^2}{(x+1)^2}f(1+\frac{1}{x}) = \frac{x^2}{(x+1)^2}(f(1)+f(\frac{1}{x})) = \frac{x^2}{(x+1)^2}(1+\frac{1}{x^2}f(x))$.

On the other hand, $f(\frac{x}{x+1}) = f(1-\frac{1}{x+1}) = f(1)+f(-\frac{1}{x+1}) = 1-f(\frac{1}{x+1}) = 1-\frac{1}{(x+1)^2}f(x+1) = 1-\frac{1}{(x+1)^2}(f(x)+1)$.

By equating both statements we get: $\frac{x^2}{(x+1)^2}(1+\frac{1}{x^2}f(x)) = 1-\frac{1}{(x+1)^2}(f(x)+1)$. Finally, $f(x) = x$.

7. Solve $(1+x^2)f'(x)+xf(x) = 1$.

Solution. At first, let us denote $y = f(x)$.

Divide by $\sqrt{x^2+1}$ and get $\frac{1}{\sqrt{x^2+1}} = y'\sqrt{x^2+1}+y\frac{x}{\sqrt{x^2+1}} = (y\sqrt{x^2+1})'$.

By taking the integral of both parts we get: $y\sqrt{x^2+1} = \ln(x+\sqrt{x^2+1})+C$. Thus, $f(x) = \frac{\ln(x+\sqrt{x^2+1})+C}{\sqrt{x^2+1}}$.

8. Find all points $P = (r,0)$ on the horizontal axis with $r\in\mathbb{Q}$, such that the distances from P to the vertices of the square $(\pm1,\pm1)$ are all rational.

Solution. We know that $(r-1)^2+1 = w_1^2$ and $(r+1)^2+1 = w_2^2$, where $w_1,w_2\in\mathbb{Q}$. By multiplying both equalities $w^2 = w_1^2w_2^2 = r^4+4$. Multiplying by the denominator of r, we get $x^4+4y^4 = z^2$.

A natural number is called **squarish**, if it is of the form n^2 or $2n^2$, where $n \in \mathbb{N}$.

We will prove the following lemma: If (a, b, c) is a Pythagorean triple and at least two numbers are squarish, then either $a = 0$ or $b = 0$.

From the lemma we can conclude that in our case $(x^2, 2y^2, z)$ either $x = 0$ or $y = 0$. By that $r = 0$ and $w_1 = w_2 = \sqrt{2}$, thus, not rational.

Now comes the proof of the lemma. Suppose we have $a^2 + b^2 = c^2$, none of them are zeros and two of them are squarish. Also, take the triple that has the smallest c. All numbers are pairwise co-prime, since if p divides two of them, then p^4 divides all of them (except for the case $p = 2$). By dividing a, b and c by p^2 (or by 2) we get a smaller triple.

It is known that there exist m and n, such that $c = n^2 + m^2$, $b = 2mn$ and $a = m^2 - n^2$, where b is even and a, c are odd.

Obviously, n and m are co-prime. If b is squarish then m and n are both squarish and at least one, a or c, is squarish. However, both of them are odd and, thus, squares. So, we get $d^2 = m^2 \pm n^2$ which is a smaller Pythagorean triple with two squarish numbers.

Otherwise, a and c are squarish and, thus, squares. Thus, (ac, n^2, m^2) is a smaller Pythagorean triple with two squarish numbers.

9. Find $\det \begin{pmatrix} a & b & b & b & b \\ a & c & d & d & d \\ a & c & e & f & f \\ a & c & e & g & h \\ a & c & e & g & i \end{pmatrix}$.

Solution. If a is zero, then the determinant is also zero.

If $b = c$ then the first two columns are linearly dependent, so the matrix is linearly dependent.

If $d = e$ then the first three columns are linearly dependent, thus, the matrix is also linearly dependent.

The same goes for $f = g$ and $h = i$.

Thus, the determinant is divisible by a, $c - b$, $e - d$ $g - f$ and $i - h$. By that, the determinant is equal to $Ca(c - b)(e - d)(g - f)(i - h)$. The only thing is left is to find a constant C.

For that we make a substitution $b = d = f = h = 0$, and the determinant becomes equal to $a \cdot c \cdot e \cdot g \cdot i$. Giving us that $C = 1$.

10. Let $P(x) = 2x^3 - 3x^2 + 2$, $A = \{P(n) \,|\, n \in \mathbb{N} \cup \{0\}, n \leq 1999\}$, $B = \{p^2 + 1 \,|\, p \in \mathbb{N} \cup \{0\}\}$ and $C = \{q^2 + 2 \,|\, q \in \mathbb{N} \cup \{0\}\}$. Prove that the sets $A \cap B$ and $A \cap C$ have the same number of elements.

Solution. Observe that $P(x) = (x - 1)^2(2x + 1) + 1$. We want it to be equal to $p^2 + 1$. Thus, either $x = 1$ or $2x + 1 = (2k + 1)^2$ (hence, $x = 2k^2 + 2k$). Since, $2 \cdot 31^2 + 2 \cdot 31 < 1999 < 2 \cdot 32^2 + 2 \cdot 32$, we derive that $A \cap B = \{P(x) \,|\, x = 2k^2 + 2k, 0 \leq k \leq 31\} \cup \{P(1)\}$.

On the other hand, $P(x) = x^2(2x - 3) + 2$. We want it to be equal to $q^2 + 2$. Thus, either $x = 0$ or $2x - 3 = (2k + 1)^2$ (hence, $x = 2k^2 + 2k + 2$). By the same reasoning, we get that $A \cap C = \{P(x) \,|\, x = 2k^2 + 2k + 2, 0 \leq k \leq 31\} \cup \{P(0)\}$.

Thus, both sets contain 33 elements.