**1.** Let $f$ be a real valued differentiable function on $(0,\infty)$ satisfying $|f(x)| \le 2$ and $f(x)f'(x) \ge \sin x$ for $x \in (0,\infty)$. Does $\lim_{x\to\infty} f(x)$ exists?

**Theorem** (Darboux's Theorem)**.** *Let $I$ be a closed interval and $f : I \to \mathbb{R}$ a real-valued differentiable function. Then $f'$ has the* intermediate value property*: If $a < b$ are points in $I$, then for every $y$ between $f'(a)$ and $f'(b)$, there exists an $x$ in $(a,b)$ such that $f'(x) = y$.*

**2.** Prove Darboux's Theorem.
**3.** Let $f : [a,b] \to [a,b]$ be a continuous function which is differentiable on $(a,b)$ and $f(a) = a$, $f(b) = b$. Prove that there exist two distinct points $x_1$ and $x_2$ in $(a,b)$ such that $f'(x_1)f'(x_2) = 1$.
**4.** Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(x) \le f'\left(x + \frac{1}{n}\right)$ for all $x \in \mathbb{R}$ and all positive integers $n$. Prove that $f'$ is continuous.
**5.** A function $f : \mathbb{R}^2 \to \mathbb{R}$ is called *olympic* if for $n \ge 3$ and any $A_1, \ldots, A_n \in \mathbb{R}^2$ distinct points such that $f(A_1) = \ldots = f(A_n)$, the points $A_1, \ldots, A_n$ are the vertices of a convex polygon.

Let $P \in \mathbb{C}[X]$ be non-constant. Prove that $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = |P(x+iy)|$ is olympic if and only if all the roots of $P$ are equal.
**6.** The continuous function $f : \mathbb{R} \to \mathbb{R}$ has the property that for any $a$, the equation $f(x) = f(a)$ has only finite number of solutions. Prove that there exist real numbers $a$ and $b$ such that the set of $x$ with $f(a) \le f(x) \le f(b)$ is bounded.
**7.** Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that $f(a)f(b) < 0$. Prove that for all integers $n \ge 3$, there exists an arithmetic progression $x_1 < \ldots < x_n$ such that $f(x_1) + f(x_2) + \ldots + f(x_n) = 0$.
**8.** Prove that there are no differentiable functions $f : \mathbb{R} \to \mathbb{R}$, such that $f'(x) - f(x) = \begin{cases} \sin x, & x \in (-\infty, 0) \\ \cos x, & x \in [0,\infty) \end{cases}$.
**9.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $a < b \in f(\mathbb{R})$. Prove that there exists an interval $I$ such that $f(I) = [a,b]$.
**10.** Let $f : [0,1] \to [0,\infty)$ be a continuous function such that $f(0) = f(1) = 0$ and $f(x) > 0$ for $0 < x < 1$. Show that there exists a square with two vertices in the interval $(0,1)$ on the $x$-axis and the other two vertices on the graph of $f$.

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