1. Consider $f(x) = 2\sin\frac{x}{2}$.

2. If y is equal to $f'(a)$ or $f'(b)$ we are done. Otherwise, consider $\phi(t) = f(t) - yt$. Suppose that $f'(a) > y > f'(b)$.

Since ϕ is continuous on the closed interval $[a,b]$ it attains its maximum on $[a,b]$ by the extreme value theorem. However, it cannot be achieved at a and b, since $\phi'(a) = f'(a) - y > 0$ and $phi'(b) = f'(b) - y < 0$.

Therefore, ϕ must attain its maximum value at $x \in (a,b)$. Hence, by Fermat's theorem, $\phi'(x) = 0$, i.e., $f'(x) = y$.

3. If there exist two points s and t such that $f'(s) < 1 < f'(t)$. Then by Darboux's theorem for each $y \in [f'(s), f'(t)]$ there exists x in $[s,t]$ such that $f'(x) = y$. Thus, we can find two satisfying points x_1 and x_2. Otherwise, we have $f'(x) \leq 1$ on (a,b) or $f'(x) \geq 1$ on (a,b). That contradicts with $f(a) = a$ and $f(b) = b$.

4. Consider the following function $f_n(x) = n\left(f\left(x + \frac{1}{n}\right) - f(x)\right)$. By the condition f_n has a non-negative derivative, thus it is increasing. Thus, for $x < y$ $f_n(x) \leq f_n(y)$. By taking $n \to \infty$, we conclude that $f'(x) \leq f'(y)$. Thus, f' is increasing and by Darboux's theorem we conclude that f' is continuous.

5. If $P(z) = a(z - z_0)^k$ and $f(A_1) = \ldots = f(A_n)$, then all A_i are on a circle of center z_0, and thus are the vertices of a convex polygon.

Now, suppose that there exist two different roots z_1 and z_2 of P such that $|z_1 - z_2|$ is minimal. Consider the line between z_1 and z_2, and let $z_3 = \frac{z_1+z_2}{2}$. Denote by s_1 and s_2 be the half lines determined by z_3. By the minimality of $|z_1 - z_2|$, $f(z_3)$ has to be greater than zero.

Also, since $\lim\limits_{|z|\to\infty, z\in s_1} f(z) = \lim\limits_{|z|\to\infty, z\in s_2} f(z) = \infty$, by the Intermediate Value Theorem there exists $z_4 \in s_1$ and $z_5 \in s_2$ such that $f(z_3) = f(z_4) = f(z_5)$.

6. First, we prove that f has limit at ∞ and $-\infty$. Suppose the opposite, then there exists two sequences a_n and b_n such that $\lim\limits_{n\to\infty} f(a_n) = a < b = \lim\limits_{n\to\infty} f(b_n)$. This implies that there exists c such that $f(a_n) < c < f(b_n)$ for all big n. By the Intermediate Value Theorem there exists an infinite sequence c_n such that $f(c_n) = c$. Contradiction.

Now, let $l_1 = \lim\limits_{x\to\infty} f(x)$ and $l_2 = \lim\limits_{x\to-\infty} f(x)$. There are two cases: the first one is $l_1 \neq l_2$. Suppose that $l_1 < l_2$. Consider $l_1 < a < b < l_2$. There are numbers A and B such that $f(x) < a$ for $x < A$ and $f(x) > b$ for $x > B$. Thus, clearly $f(x) \in [a,b]$ only happens for $x \in [A,B]$, which is bounded.

The second case is $l_1 = l_2 = l$. We can choose either $a < b < l$ or $l < a < b$. using the same argument as in the first case we are done.

7. Assume that $f(a) < 0$ and $f(b) > 0$. There exists ϵ such that $f(x) < 0$ for $x \in [a, a+\epsilon]$ and $f(x) > 0$ for $x \in [b-\epsilon, b]$. Let us consider two arithmetic progressions: $a_1 < a_2 < \ldots < a_n \in [a, a+\epsilon]$ and $b_1 < b_2 < \ldots < b_n \in [b-\epsilon, b]$.

Define $F(t) = \sum\limits_{k=1}^{n} f((1-t)a_k + tb_k)$. $F(0) = \sum\limits_{k=1}^{n} f(a_k) < 0$ and $F(1) = \sum\limits_{k=1}^{n} f(b_k) > 0$. Thus, by the Intermediate Value Theorem there exists τ with $F(\tau) = 0$.

8. Suppose such a function exists. Consider $g(x) = e^{-x} f(x)$. Its derivative has Intermediate Value Property by Darboux's Theorem. However, it has a discontinuity at zero: $\lim\limits_{x\to0-} = \lim\limits_{x\to0} e^{-x}\sin x = 0$ and $\lim\limits_{x\to0+} = \lim\limits_{x\to0} e^{-x}\cos x = 1$. Contradiction.

9. There exist p and q such that $f(p) = a$ and $f(q) = b$. Let $p < q$.

The set of $f(x) = a$ on $[p,q]$ is bounded thus it has a supremum α. The set of $f(x) = b$ on $[\alpha, q]$ is bounded thus it has an infimum β. We claim that $f([\alpha,\beta]) = [a,b]$.

Obviously, due to the Intermediate Value Theorem $f([\alpha,\beta]) \supseteq [a,b]$. Suppose that $f(x) < a$ for some $x \in [\alpha,\beta]$. Thus, by the continuity there exists $y \in [x,\beta]$ with $f(x) = a$. Contradtiction, since α is a supremum. The same goes if $f(x) > b$.

10. $A = \{x \in [0,1] : x + f(x) \geq 1\}$ is non-empty and has an infimum. Let it be c. Since $0 + f(0) < 1$ and f is continuous, the inequality $x + f(x) < 1$ holds in a neighbourhood of the origin, therefore $c > 0$. To summarize: there is a $c \in (0,1]$ such that $c + f(c) = 1$ and $x + f(x) < 1$ for all $x \in [0,c)$.

Since f is continuous, it attains extremum $f(d)$ on $[0,1]$ where $d \in (0,1)$. Consider $g(x) = x + f(x)$. Since $g(0) = 0 < d < 1 = c + f(c) = g(c)$, there exists α with $g(\alpha) = d$.

Now consider $h(x) = f(x + f(x)) - f(x)$ on $[0,c]$. We have $h(\alpha) = f(\alpha + f(\alpha)) - f(\alpha) = f(d) - f(\alpha) \geq 0$, $h(c) = f(c + f(c)) - f(c) = f(1) - f(c) = -f(c) \leq 0$, and $h(d) = f(d + f(d)) - f(d)$ only if $d \leq c$. Thus, by the Intermediate Value Theorem $h(\gamma) = 0$ with $\gamma \in [\alpha, c]$. However, we cannot be sure that $\gamma < 1$. If $\gamma = 1$, then $c = 1$, and thus $d < c$ and we can choose another γ from $[\alpha, d]$. Thus, we found a square.